Monotone control of queueing networks

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Received 30 October 1991; revised 22 May 1992

This paper uses submodularity to obtain monotonicity results for a class of Markovian queueing network service rate control problems. Nonlinear costs of queueing and service are allowed. In contrast to Weber and Stidham [14], our monotonicity theorem considers arbitrary directions in the state space (not just control directions), arrival routing problems, and certain uncontrolled service rates. We also show that, without service costs, transition-monotone controls can be described by simple control regions and switching functions. The theory is applied to queueing networks that arise in a manufacturing system that produces to a forecast of customer demand, and also to assembly and disassembly networks.

Keywords: Control of queues; dynamic programming; submodularity; monotone policies; make-to-stock queues.

1. Introduction

The optimal control of arrival and service rates in networks of queues has been the topic of considerable research (see Stidham [11,12] and the references therein). Most studies have dealt with specific network structures or systems with only two queues. Even in these special cases and with memoryless arrival and service processes, one usually must resort to numerical dynamic programming techniques to compute the optimal control. Alternatively, a certain form of control can be analyzed; examples include end-to-end controls for communication systems, and kanban and base stock controls for manufacturing systems.

Another fruitful line of research has been to study the *structure* of optimal policies. In many cases, the optimal policy has been shown to have intuitive monotonicity properties (see, for example, Rosberg et al. [10], Beutler and Teneketzis [1], and Hajek [6]). These proofs all use submodularity and are problem-specific. Weber and Stidham [14] use submodularity to prove *transition* *monotonicity:* service completion at one station cannot reduce the optimal service rate at another station. Their result applies to a class of network topologies, which are assumed to be Markovian with all service rates controllable, controllable and uncontrollable arrival rates, and discounted or long-run average costs. In a series system with service rate control, service costs and convex holding costs, for example, they show that upon the addition of a customer to a queue, the optimal service rates at that queue and at downstream queues do not decrease and the optimal service rates upstream do not increase.

This paper generalizes the monotinicity results of Weber and Stidham [14] in several straightforward but useful ways. First, they only consider monotonicity with respect to control directions in the state space. Although other directions can often be addressed by adding fictitious controls, we take the straightforward approach of considering arbitrary directions; the production/inventory model that motivated our research uses this generalization. Second, arrival routing problems, such as Hajek's [6] two interacting queues, are addressed using controls that choose between two transitions. Finally, they treat uncontrolled transitions that (i) are state-independent, i.e., arrivals or (ii) can be replaced by controlled transitions for which the maximum rate is always optimal (see [14, p. 217]). We consider other uncontrolled transitions, including the service rates in [6] and the downstream server in a series queue.

The following assumptions are made to establish monotonicity:

(1) holding costs satisfy a submodularity condition (related to convexity);
(2) the boundaries of the state space satisfy a geometric condition: only one control (or other vector used as a monotone direction) can cross each boundary in each direction;
(3) controls that choose between two directions must have one direction that never leaves the state space; and
(4) service and arrival rates are controllable to zero, state-independent, or satisfy an additional condition (preserving submodularity).

The theory is powerful in the sense that it can be used to easily prove monotonicity for a variety of queueing control problems. We have used it to reproduce some of the two-station monotonicity results of Rosberg et al. [10] and Hajek [6].

Another extension of transition monotonicity is the recent paper by Glasserman and Yao [5]. They replace (2), which is a lattice condition on the state space, with a weaker join semi-lattice condition plus some other conditions on the holding costs, use *super-* and submodularity, and consider controls that choose between two directions or idleness. Their results could potentially be applied to routing decisions within a network or to server allocation (scheduling) problems, where our theorem does not apply. However, neither their result nor ours reproduces the monotonicity proven by Hajek for his allocation problem.

The monotonicity theorem is applied to several problems (some of which can be treated using Weber and Stidham [14]). Of particular interest is the following queueing network that arises in a manufacturing setting. Items pass through a sequence of exponential single-server queues, each of which represents a stage of production. Upon exiting the final stage, items are placed in finished goods inventory that is used to meet a Poisson demand. Demand that cannot be met from inventory is backordered and met by the next available finished item. Production and holding costs are incurred at each stage, as well as finished goods backorder costs. The problem is to control the production rates to minimize discounted or average costs. This system is known as a *make-to-stock* queueing system, or in the manufacturing literature as a production/inventory system (see, for example, Buzacott et al. [2]). For a series system, we show that upon the addition of an item between stages, the production rates downstream do not decrease and the production rates upstream do not increase; also, a demand does not decrease production rates.

We also give some simplifying characterizations of the optimal policy under transition monotonicity. It is well-known that, in the absence of service costs, these problems have all-or-nothing (bang-bang) optimal policies. We show that they are characterized by switching functions with certain slope restrictions that bound a region in which all servers work at full capacity. When the system state reaches a boundary, one or more servers turn off. This simplification should ease the task of numerically computing optimal policies. Switching functions are also discussed in Hajek [6], Stidham [11,12], and Chen et al. [3] for specific two-station networks. Developing numerical techniques that exploited these switching functions to compute optimal policies would be a useful area for further research.

We define a general queueing control problem in section 2, then present monotonicity theorems in section 3. Transition-monotone policies are further characterized in section 4, some generalizations covered in section 5, and applications to make-to-stock queueing networks and other examples discussed in section 6. We will use the term increasing (decreasing) in the weaker sense of nondecreasing (nonincreasing) and denote a unit vector whose ith component is one by e_i.

2. Problem description

We are primarily concerned with arrival rate, service rate, and routing control problems for queueing networks. However, the method being used to obtain monotonicity results applies to a broader class of Markov decision processes, so we will follow Weber and Stidham and use a more general notation. To accommodate certain routing problems, we define controls that choose between two transitions. If, instead, the rate of each transition is controlled independently, set $d_i^0 = 0$ in what follows. Consider a continuous time Markov decision process with state $x = (x_1, \dots, x_m) \in X \subseteq Z^m$. The decision space is $\{\mu = (\mu_1, \dots, \mu_q): 0 \leqslant \mu_i \leqslant \bar{\mu}_i\}$ and, given a control μ in state x, the transition $x \to x + d_i^1$ occurs at rate μ_i while the transition $x \to x + d_i^0$ occurs at rate $\bar{\mu}_i - \mu_i$, for $i \in \mathscr{C} = \{1, \dots, q\}$. Let $d_i = d_i^1 - d_i^0$; the control μ_i pushes in the direction d_i. Because the system is memoryless, an optimal control need only depend on the current state x; we denote this control $\mu(x)$. Also, if $x + d_i^1 \notin X$, then $\mu_i(x) = 0$.

Let $d_{q+1}, \dots, d_{q+p}$ be the uncontrolled transitions with index set $\mathscr{U} = \{q+1, \dots, q+p\}$. To prevent transitions out of the state space, define the transition function $D_i x = x + d_i$ if $x + d_i \in X$ and $D_i x = x$ otherwise; D_i occurs at the constant rate λ_i. The cost rate is a separable, nonnegative function of the state,

$$h(x) = \sum_{k=1}^{m} h_k(x_k),$$

with certain restrictions defined later, plus a continuous, separable function of the control,

$$c(\mu) = \sum_{i \in \mathscr{C}} c_i(\mu_i).$$

The functions c_i are assumed to be convex; if they are not, an equivalent optimization problem results when c_i is replaced by its convex lower envelope on $[0, \bar{\mu}_i]$. The objective is to minimize the expected discounted cost. An average cost criterion will be considered in section 5.

In a queueing network control problem, x_i is the queue length (including customers in service) at queue i; $X = Z_+^m$; $\mathscr{C}$ consists of a customer moving from queue j to queue k ($d_i^1 = e_k - e_j$, $d_i^0 = 0$), a customer departing the network from queue j ($d_i^1 = -e_j$, $d_i^0 = 0$), controlled arrivals at queue j ($d_i^1 = e_j$, $d_i^0 = 0$), and routing arrivals to queue j or k ($d_i^1 = e_k$, $d_i^0 = e_j$); μ_i is a service, arrival, or routing rate control; $\mathscr{U}$ may contain uncontrolled arrivals or departures; $h_k(x_k)$ is the cost of holding x_k customers at queue k; and $c_i(\mu_i)$ is the cost of service (or arrivals) at rate μ_i.

In make-to-stock queueing networks, one of the state variables, say x_n, measures finished goods inventory and is allowed to be negative, representing backorders. The uncontrolled transitions are demands which decrease the finished goods inventory. The controls μ_i are arrival and service rates with costs c_i; the costs h_k are holding costs at each station $k \neq n$ and h_n is the finished good holding and backorder cost. A complete description is given in section 6.

For the infinite-horizon problem with discount rate $\alpha > 0$, the minimum expected cost is

$$V(x) = E_x \int_0^\infty e^{-\alpha t} [h(x(t)) + c(\mu(t))] \, dt,$$

where E_x denotes expectation given the initial state $x(0) = x$. We will uniformize the process as in Lippman [9] by defining the potential event rate

$$\Lambda = \sum_{i \in \mathscr{C}} \bar{\mu}_i + \sum_{i \in \mathscr{U}} \lambda_i.$$

Let $V_n(x)$ be the minimum n-stage expected discounted cost for the embedded discrete-time Markov decision process. Then V_n is well-defined and satisfies the dynamic programming equation

$$\begin{aligned}(\alpha + \Lambda)V_{n+1}(x) = h(x) &+ \sum_{i \in \mathscr{U}} \lambda_i V_n(D_i x)\\ &+ \sum_{i \in \mathscr{C}} \min_{0 \leqslant \mu_i \leqslant \bar{\mu}_i} \{c_i(\mu_i) + \mu_i V_n(x + d_i^1)\\ &+ (\bar{\mu}_i - \mu_i)V_n(x + d_i^0)\},\end{aligned} \tag{2.1}$$

where we define $V_0(x) = 0$ and $V_n(x) = \infty$, $x \notin X$.

3. General monotonicity results

Monotonicity results will be obtained by investigating the effect on the optimal control of moving in certain directions in the state space.

DEFINITION

The control $\mu(x)$ is **D**-*monotone* if

$$\mu_j(x) \leqslant \mu_j(x + d_i)$$

for all controls $j \in \mathscr{C}$ and direction vectors $d_i \in \mathbf{D}$ such that $d_i \neq d_j$.

We will only consider direction sets **D** that contain all of the control directions d_i, $i \in \mathscr{C}$. If $\mathbf{D} = \{d_i : i \in \mathscr{C}\}$ and $d_i^0 = 0$ (i.e., **D** is the set of control transitions), we will say that $\mu(x)$ is *control-monotone* (this is the case considered in Weber and Stidham [14]); if $\mathbf{D} = \{d_i : i \in \mathscr{C} \cup \mathscr{U}\}$ and $d_i^0 = 0$ (i.e., **D** is the set of all transitions), we will say that $\mu(x)$ is *transition-monotone*. For a queueing system with uncontrolled arrivals and controlled servers, control monotonicity means that after a service completion the service rate of other servers increases; transition monotonicity means that the service rate of all servers also increases after an arrival.

Remark

Under transition monotonicity, μ_j is increasing in every other transition. If $\{x(t),\ t \geqslant 0\}$ is an irreducible Markov chain under some control $\mu(x)$, then we can write $-d_j = \sum_{i=i_1}^{i_n} d_i$ for some sequence of transitions not including d_j; hence, μ_j is decreasing in d_j. This monotonicity result might be called a law of diminishing returns. Weber and Stidham use this reasoning for a cycle of queues.

Considering the minimization in (2.1), if

$$V_n(x + d_i^1) - V_n(x + d_i^0) \geqslant V_n(x + d_i^1 + d_j) - V_n(x + d_i^0 + d_j), \tag{3.1}$$

for all $i \in \mathscr{C}$, $d_i \neq d_j \in \mathbf{D}$, and x such that all four points are in X, then the future cost of control μ_i decreases when transition d_j occurs. Because c_i is convex, a **D**-monotone optimal control must exist at stage $n + 1$. For simplicity, change x in (3.1) and write

$$V_n(x + d_i) - V_n(x) \geqslant V_n(x + d_i + d_j) - V_n(x + d_j). \tag{3.2}$$

Since $V = \lim_{n \to \infty} V_n$, if (3.2) holds with V_n replaced by V then **D**-monotonicity also holds for the infinite-horizon problem. Condition (3.2) is an example of submodularity of a function on a lattice (Topkis [13]).

DEFINITION

The function $f : X \to R$ is *submodular* w.r.t. **D** on X if

$$f(x + d_i) + f(x + d_j) - f(x) - f(x + d_i + d_j) \geqslant 0, \tag{3.3}$$

for all $d_i \neq d_j \in \mathbf{D}$ and x such that all four points are in X.

The following lattice condition, introduced in Weber and Stidham [14], will be used to guarantee that (3.2) holds sufficiently close to the boundary of X.

DEFINITION

D is *compatible* with X if for all $d_i \neq d_j \in \mathbf{D}$, $x + d_i$ and $x + d_j \in X$ implies x and $x + d_i + d_j \in X$.

For most queueing control problems, we can think of X as the discrete equivalent of a polyhedral set and give a simple interpretation of compatibility: only one d_i can cross each bounding hyperplane in each direction. Let $\mathbf{D}(x) = \{d \in \mathbf{D} : x + d \in X\}$ be the set of feasible directions from state x. If **D** is compatible with X and f is submodular w.r.t. **D** on X, we will say that f is submodular w.r.t. $\mathbf{D}(x)$, since (3.3) holds for all $x \in X$ and $d_i \neq d_j \in \mathbf{D}(x)$.

Weber and Stidham establish submodularity of V by showing that submodularity is preserved under value-iteration. The key step is the following lemma, which is proven in the appendix (their proof is very similar but does not include choice controls).

LEMMA 1

If **D** contains all control directions d_i, $i \in \mathscr{C}$,

(i) $f(x)$ is submodular w.r.t. $\mathbf{D}(x)$, with $f(x) = \infty$, $x \notin X$,

(ii) **D** is compatible with X, and

(iii) d_i^0 is always feasible, i.e., $x + d_i^0 \in X$ whenever $x \in X$, then for each d_k, $k \in \mathscr{C}$,

$$g(x) = \min_{0 \leqslant \mu \leqslant \bar{\mu}_k} \{c_k(\mu) + \mu f(x + d_k^1) + (\bar{\mu}_k - \mu) f(x + d_k^0)\}$$

is also submodular w.r.t. $\mathbf{D}(x)$.

Submodularity must also be preserved by uncontrolled transitions (the first sum in eq. (2.1). One type of transition that preserves submodularity is a state-independent transition, $D_i x = x + d_i$ for all $x \in X$, i.e., Poisson arrivals. Some other transitions also work, depending on the geometry of $\mathbf{D}$ and X; see section 6 for examples.

DEFINITION

The uncontrolled transition D_i, $i \in \mathscr{U}$, preserves submodularity w.r.t. $\mathbf{D}(x)$ if $f(D_i(x))$ is submodular whenever $f(x)$ is submodular.

Monotonicity now follows easily from the lemma.

THEOREM 1 (Monotonicity)

If $\mathbf{D}$ contains all control directions d_i, $i \in \mathscr{C}$,
(i) $h(x)$ is submodular w.r.t. $\mathbf{D}(x)$,
(ii) $\mathbf{D}$ is compatible with X,
(iii) d_i^0 is always feasible, and
(iv) all uncontrolled transitions preserve submodularity w.r.t. $\mathbf{D}(x)$,
then there exists a $\mathbf{D}$-monotone optimal control.

Proof

First show that $V_n(x)$ is submodular w.r.t. $\mathbf{D}(x)$ using induction on n. Initially $V_0(x) = 0$ is submodular. In (2.1), $h(x)$ is submodular by (i), the first sum by (iv), and the second sum by the lemma and the inductive hypothesis. Since $V_n \to V$, it follows that V is submodular w.r.t. $\mathbf{D}(x)$ and a $\mathbf{D}$-monotone optimal control exists. □

Note that the inductive proof establishes monotonicity for finite and infinite-horizon problems.

Weber and Stidham's monotonicity result is equivalent to the following specialization of our theorem.

COROLLARY (Control Monotonicity)

If $\mathbf{D} = \{d_i : i \in \mathscr{C}\}$, $d_i^0 = 0$ for all $i \in \mathscr{C}$, and (i), (ii), and (iv) above, then there exists a control-monotone optimal control.

The generalization of adding other directions to $\mathbf{D}$ is very useful in some problems because it strengthens the monotonicity result; the strongest way to use this generalization is to consider one additional direction at a time. For systems with uncontrolled transitions, transition monotonicity (if it holds) tends to be a much stronger result than control monotonicity. The next section describes some of the useful implications of transition monotonicity. Choice controls with $d_i^0 \neq 0$ are useful in some arrival routing problems. We have not been able to establish monotonicity without condition (iii); with this condition, the theorem does not apply to problems with server allocation or routing between servers.

Remark

If **D**-monotonicity holds with e_k or $-e_k$ in **D**, it can be used to establish monotonicity with respect to the x_k axis. Depending on the geometry of the problem, it may be possible to infer state space monotonicity from other direction sets. **D**-monotonicity establishes that the optimal control for a transition d_i increases after any other transition in **D**. It will also increase after any sequence of transitions in **D**; hence, the convex cone C_i of $\mathbf{D} \setminus \{d_i\}$ is a set of directions in which μ_i is increasing and the negative convex cone C_i^- is a set of directions in which μ_i is decreasing. Figure 1 shows these cones for the two-stage production/inventory system of section 6 under transition monotonicity, where $\mathbf{D} = \{d_1, d_2, d_3\}$. If the vector e_k, giving the direction of the x_k-axis, lies within C_i or C_i^-, then monotonicity holds with respect to the x_k coordinate. For example, the control d_1 (and d_2) in fig. 1 is monotone with respect to both x_1 and x_2. Control monotonicity corresponds to the convex cones generated by smaller sets **D**. If only control monotonicity is established in fig. 1, the convex cone of μ_1 is the single direction d_2, which is not sufficient to establish state space monotonicity.

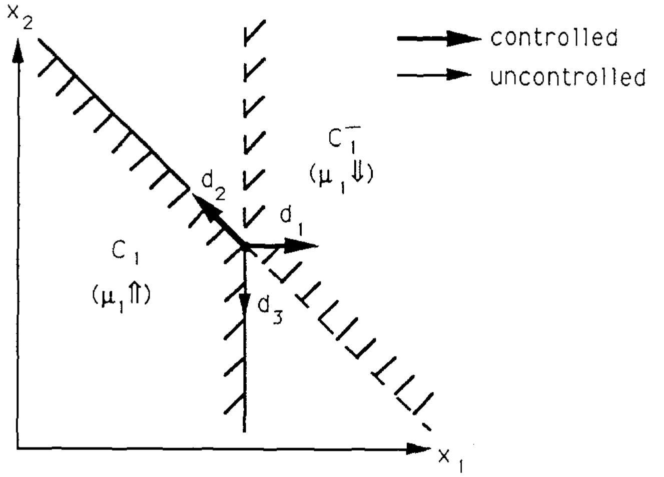

Fig. 1. Increasing and decreasing directions for the control μ_1.

4. Control regions and switching functions

It is well known that if the service cost c_i in (2.1) is linear, then an all-or-nothing (bang-bang) control is optimal, with $\mu_i(x) = 0$ or $\bar{\mu}_i$. Bang-bang, transition-monotone controls can be characterized very simply as a control region. Let $S_i = \{x \in X : \mu_i(x) = \bar{\mu}_i\}$, $S_i' = \{x + d_i : x \in S_i\}$, and $B_i = S_i' \setminus S_i$ for $i \in \mathscr{C}$. In queueing terminology, S_i is the region where server i is on. As illustrated in fig. 2, B_i contains points on the boundary of S_i that are reached by one transition d_i. For any $x \in S_i$, consider the cone C_i for control μ_i with vertex x. Since μ_i is on at x, it must be on in all of C_i (similarly, if μ_i were off at x, it would be off in all of C_i^-); i.e., $C_i \cap X \subseteq S_i$. But all transitions other than d_i lie in C_i, so they cannot cause the system state to leave S_i. By a translation argument, these transitions cannot leave S_i' either. Under such a control, the system never leaves the *control region* $S' = \bigcap_{i \in \mathscr{C}} S_i'$. This region consists of an interior $S = \bigcap_{i \in \mathscr{C}} S_i$ in which all controls are on and boundaries in which one or more controls are off. The system can only enter a boundary in which μ_i is off through the transition d_i.

The control region can usually be defined using a *switching function* for each control μ_i. Hajek proves that switching functions exist for a system of two interacting queues; we give sufficient conditions for the existence of switching functions in general **D**-monotone systems. If the direction e_k lies in the cone C_i or C_i^- for control μ_i, then S_i can be defined by a switching function $s_i(x^{(k)})$, where $x^{(k)} = (x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_m)$, such that $\mu_i(x) = \bar{\mu}_i$ if and only if

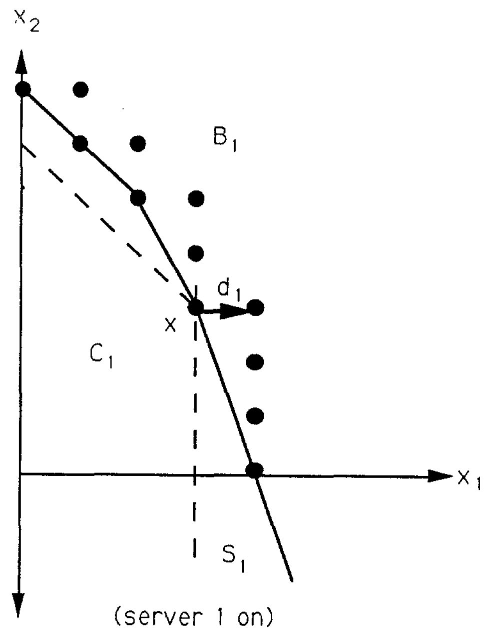

Fig. 2. The cone C_1 (to the left of the dashed line) is contained in the interior region S_1 (to the left of the solid line); B_1 contains the points outside of S_1.

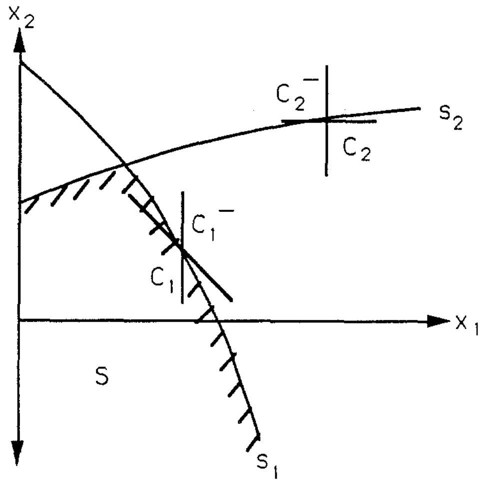

Fig. 3. Switching functions and the interior region.

$x_k \geqslant (\leqslant) s_i(x^{(k)})$. In this case, the interior S is simply a subset of X bounded by the switching functions s_i. Moreover, the function s_i must lie between C_i and C_i^- for the control μ_i. Figure 3 gives an idealization of these regions (the s_i are actually step functions). In queueing applications, these controls turn a server off when a boundary is reached; such controls have been analyzed on a continuous state space as reflected Brownian motion (Harrison [8]).

5. Some generalizations

Although compatibility is a useful condition because it is easily verified, other conditions can be used in the theorem. If it is possible to extend the cost function h onto all of Z^n in a way that preserves the submodularity of h and makes h much larger at $y \notin X$ than at nearby $x \in X$, then h acts as a barrier and the optimal control for the unbounded problem ($V_0(x) = 0$ for all x) will be feasible for the original problem. Since there are no boundaries, compatibility is not needed to prove the lemma for the unconstrained problem and establish transition monotonicity. Hajek uses a similar extendibility approach to prove monotonicity in a two-station network. However, extendibility of the cost function is at least as strong a condition; it implies compatibility, as the following argument shows.

To formalize the concept of nearby values, let

$$n_{\sup}(y) = \max\{h(x) : x \in X;\ x \pm d = y,\ d \in \mathbf{D}\},$$

$$n_{\inf}(y) = \min\{h(x) : x \in X;\ x \pm d_i = y \text{ or } x \pm d_i \pm d_j = y,\ d_i,\ d_j \in \mathbf{D},\ i \neq j\}.$$

If the set used in its definition is empty, then set $n_{\sup}(y)$ or $n_{\inf}(y)$ to zero. Suppose that h is submodular with respect to $\mathbf{D}$ on X and that for any M there exists an extension $\tilde{h}: Z^m \to R$ of h that is submodular on Z^m with $\tilde{h}(x) \geqslant 2n_{\text{sup}}(x) - n_{\text{inf}}(x) + M$. Then for any controlled transition $d_i \in \mathbf{D}$ and x such that $x \in X$ and $x + d_i \notin X$, we have $\tilde{h}(x + d_i) \geqslant \tilde{h}(x) + M$, and for sufficiently large M the optimal control for the unbounded problem is $\mu_i(x) = 0$. Hence, this control is feasible (and optimal) for the original problem.

To show compatibility, suppose there exists d_i, $d_j \in \mathbf{D}$, $i \neq j$, and x such that $x + d_i$, $x + d_j \in X$ and $x \notin X$. Then

$$\tilde{h}(x + d_i) + \tilde{h}(x + d_j) - \tilde{h}(x) - \tilde{h}(x + d_i + d_j)$$
$$\leqslant n_{\text{sup}}(x) + n_{\text{sup}}(x) - \left[2n_{\text{sup}}(x) - n_{\text{inf}}(x) + M\right] - n_{\text{inf}}(x),$$

which is negative for $M > 0$. But this contradicts submodularity of $\tilde{h}$, so $x \in X$. A similar argument shows that $x + d_i + d_j \in X$.

The compatibility condition can be weakened slightly if there are no service costs. Given four points in X on which f is submodular, as in (3.3), $g(x)$ in the lemma is only submodular on these points if certain combinations of the four points obtained by adding d_k, $k \neq i$ or j, are in X (i.e., points (b), (d), (f), and (g) in fig. 4). Writing

$$g(x) = \bar{\mu}_k \min\{f(x), f(x + d_k)\},$$

a case-by-case analysis reveals that nine of the 2^4 possibilities guarantee submodularity. In contrast, compatibility allows only five of these possibilities (table 1) and must hold for any four points x, $x + d_i$, $x + d_j$, and $x + d_i + d_j$, not just those defined as (b), (d), (f), and (g) are. If $k = i$, a similar analysis shows that compatibility is as weak as possible, allowing three out of 2^2 possible combinations of points in X. However, we have not found any meaningful examples where $g(x)$ is submodular and conditions (i), (iii), and (iv) hold but compatibility does not hold.

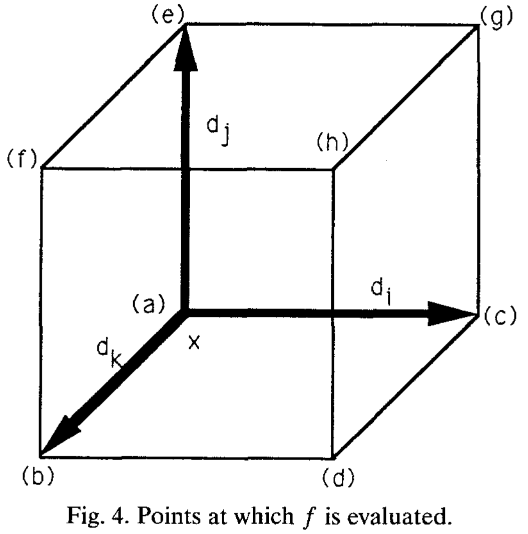

Fig. 4. Points at which f is evaluated.

We have assumed that the lower control limits are zero. Positive lower control limits $\underline{\mu}_i \leqslant \mu_i \leqslant \bar{\mu}_i$ can be modelled by introducing a duplicate uncontrolled transition $d_j = d_i$, $j \in \mathscr{U}$, $i \in \mathscr{C}$, and resetting the control limits to $\lambda_k = \underline{\mu}_i$ and $0 \leqslant \mu_i \leqslant \bar{\mu}_i - \underline{\mu}_i$. However, the duplicate transition in $\mathbf{D}$ adds the following conditions to the theorem:

(1) By submodularity, $2h(x+d_i) - h(x) - h(x+2d_i) \geqslant 0$, i.e., h is convex in the direction d_i.
(2) By compatibility, if $x + d_i \in X$ then x, $x + 2d_i \in X$; i.e., X has no boundaries in the directions $\pm d_i$.

Monotonicity can be extended to the long-run average cost setting; the proof given by Weber and Stidham applies directly. If the state space X is finite (or h is bounded), well-known methods can be applied to let $\alpha \to 0$. In practice, this is not much of a restriction because it is usually clear how to truncate the state space. Note, however, that monotonicity holds for discounted optimal controls even when the system is unstable and cannot be truncated. For unbounded h we need the following assumptions:

(1) There exists a policy with finite average cost.
(2) For all x, $y \in X$, there exists a policy under which the undiscounted cost until first passage from x to y is finite.

Table 1
Comparison of boundary conditions.

Case	Points in X	Compatible	g Submodular
1	b, d, f, h	√	√
2	d, f, h		
3	b, f, h		√
4	b, d, h		√
5	b, d, f		
6	f, h	√	√
7	d, h	√	√
8	d, f		
9	b, h		√
10	b, f		
11	b, d		
12	h	√	√
13	f		
14	d		
15	b		√
16	–	√	√

(3) The cost functions h_k are bounded below.
(4) For some ordering of the states $x \in X$, $h(x) \to \infty$.

The second assumption is a stronger form of irreducibility and the fourth guarantees that there are only finitely many good states. With these additional assumptions, the theorem holds for average cost optimal policies.

6. Applications

Tandem make-to-stock queueing systems

Consider the tandem queueing system of fig. 5 in which each stage operates as a $\cdot/M/1$ queue. Items are endogenously released into the system, pass through each stage, and then placed in a finished goods inventory that services a Poisson demand. Demand that cannot be met from inventory is backordered and recorded as a negative inventory. Unlike traditional queueing systems, this make-to-stock system allows deficits in the final "queue". One application of this model is a production/inventory system, where each stage represents a machine in a production line; this terminology will be used to discuss make-to-stock systems.

Denote the system state by $x = (x_1, \dots, x_n)$, where x_i is the number of items at stage $i+1$, $i = 1, \dots, n-1$ and x_n is the finished goods inventory. Because the supply of raw material is unlimited, there is no queueing and no state variable at stage 1. Stage i has production rate μ_i controlled between 0 and $\bar{\mu}_i$, with transitions $d_1 = e_1$ and $d_i = e_i - e_{i-1}$, $i = 2, \dots, n$. Demand is an uncontrolled transition $d_{n+1} = -e_n$ with rate λ. These transitions are illustrated in fig. 1 for a two-stage system. The state space is $X = \{x \in Z^n : x_i \geqslant 0,\ i = 1, \dots, n-1\}$. There is a work-in-process holding cost h_k after stage k (items available to stage $k+1$), $k = 1, \dots, n-1$, a finished goods holding cost h_n, and a finished goods backorder cost b; i.e.,

$$h(x) = \sum_{k=1}^{n-1} h_k x_k + h_n x_n^+ + b x_n^- . \qquad (4.1)$$

No holding cost is assessed at stage 1 because it will contain at most one item in production and none queued. The production cost $c(\mu)$ is approximately the

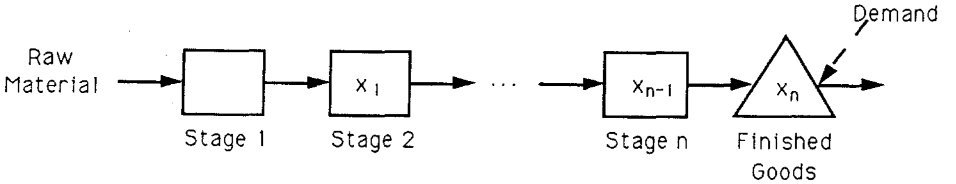

Fig. 5. A tandem make-to-stock queueing system.

same for all stable controls, and so is assumed to be zero. It is interesting to note that this tandem make-to-stock queueing system differs from a traditional tandem queueing system only in the removal of one boundary from X, the convex (as opposed to increasing) cost function, and which transition is uncontrolled. As suggested in section 1, the corollary (Weber and Stidham's result) can be applied to this system by replacing the uncontrolled demands with a controllable transition with an arbitrarily large reward for the control, so that the maximum rate will always be optimal. A more direct approach is to let $\mathbf{D}$ contain all transitions and check the following conditions.

(1) *h is submodular w.r.t.* $\mathbf{D}(x)$. The submodularity condition (3.3) clearly holds when all four points lie on the same linear segment of h. The exception involves the transitions $e_n - e_{n-1}$ and $-e_n$ at $x_n = 0$; here (3.3) is

$$h(x+e_n-e_{n-1}) + h(x-e_n) - h(x) - h(x-e_{n-1}) \geqslant 0.$$

Substituting (4.1) into the above gives

$$h(x) + h_n - h_{n-1} + h(x) + b - h(x) - h(x) + h_{n-1} = h_n + b \geqslant 0.$$

(2) $\mathbf{D}$ *is compatible with* X. The only transitions that cross the boundary $x_k = 0$, $k < n$, are $e_k - e_{k-1}$ in the positive direction and $e_{k+1} - e_k$ in the negative direction, so the geometric condition for compatibility is satisfied. A case-by-case check of compatibility can also be performed.

There are no choice controls ($d_i^0 = 0$) and Poisson demand preserves submodularity, so we conclude that there exists a transition-monotone optimal control. To determine state space monotonicity, note that

$$\begin{aligned} e_k &= d_1 + \cdots + d_k \\ &= -(d_{k+1} + \cdots + d_n + d_{n+1}). \end{aligned}$$

For $k < j$, the first equality implies that μ_j is increasing in x_k (upstream inventory); for $k \geqslant j$, the second equality implies that μ_j is decreasing in x_k (downstream inventory). Furthermore, the optimal control is bang-bang with the following switching functions: stage i is on when $x_i \leqslant s_i(x^{(i)})$ and off otherwise (recall that x_i is the inventory immediately downstream from stage i and that $x^{(i)}$ is the vector of state variables other than x_i). The rates of change of the switching functions are also constrained because the corresponding regions S_i must contain the cones C_i. In the two-stage system of fig. 3, s_1 has a slope no greater than -1 (more precisely, the step function s_i is decreasing and has no horizontal segment of length greater than one) and s_2 is increasing. It is interesting to note that the base stock policy that has been proposed for this system (see Buzacott et al. [2]) uses the limiting slopes of -1 for s_1 and zero for s_2. Similar characterizations of the optimal control of arrivals to two queues are given in Ghoneim and Stidham [4].

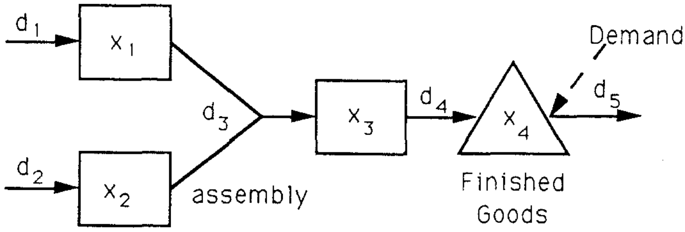

Fig. 6. An assembly make-to-stock queueing system.

Assembly and disassembly systems

For make-to-stock queueing networks with topologies that include assembly, such as in fig. 6, very similar arguments can be used. Each stage except finished goods has a boundary $x_k \geqslant 0$ which is crossed exactly once in the increasing direction and once in the decreasing direction, so compatibility holds. The cost function $h(x)$ is submodular as before. Transition monotonicity implies that μ_j is increasing in upstream inventory and inventory in other branches, and decreasing in downstream inventory. For the example in fig. 6, μ_1 is increasing in x_1 and x_2 and decreasing in x_3 and x_4; μ_3 is increasing in x_1, x_2, and x_3 and decreasing in x_4.

Reversing the above network topology gives a disassembly system, which might arise when a single production process yields multiple products. Similar arguments show that μ_j is increasing in upstream inventory and decreasing in downstream inventory and inventory in other branches. Analogous results hold for traditional queueing networks with assembly or disassembly topologies.

It should be noted that these results do not apply to systems with more than one transition in or out of a station. For example, if stations 1 and 2 in fig. 6 supply the same part to station 3, $d_3 = e_3 - e_1 - e_2$ is replaced by two transitions, $e_3 - e_1$ and $e_3 - e_2$, and compatibility is lost. A distribution network with more than one retail location supplied by a single supplier is also not compatible.

Arrival routing

The theorem applies to the arrival routing problem shown in fig. 7, which is a special case of the problem considered in Hajek [6]. In addition to uncontrolled

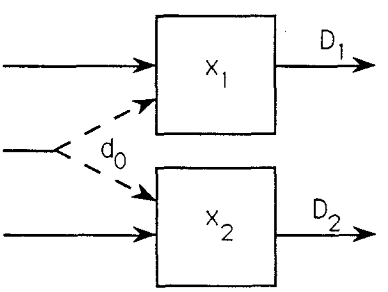

Fig. 7. A system with arrival routing.

arrivals at each queue, the choice control $d_0 = e_2 - e_1$ routes arrivals to queue 1 ($d_0^0 = e_1$) or queue 2 ($d_0^1 = e_2$). There are also uncontrolled servers, $D_i x = x - e_i$, $x_i > 0$, $i = 1, 2$. As usual, the state space is $X = R_+^2$ and $h(x)$ is a linear holding cost. Let $\mathbf{D} = \{e_2 - e_1, e_1, -e_2\}$. The only difficult condition to check is (iv). The transitions D_1 and D_2 do not preserve submodularity for general functions $f(x)$; the submodularity conditions for $f(D_i x)$ require that f be convex and increasing in the x_1 and x_2 directions at points on the boundary $x_1 = 0$ or $x_2 = 0$. Hajek [6] notes that submodularity w.r.t. $\mathbf{D}(x)$ implies convexity; a stochastic coupling argument shows that $V_n(x)$ is increasing in x_k. With these auxiliary conditions, D_1 and D_2 preserve submodularity of $V_n(x)$ and $\mathbf{D}$-monotonicity is established; namely, the optimal arrival rate at queue 2 is increasing in x_1 and decreasing in x_2.

Uncontrolled servers

The previous example illustrates that the theorem applies to some problems with uncontrolled service rates; another example is a series queue. Let x_i be the number of customers at queue i, $i = 1$ (upstream) and 2 (downstream), and $D = \{e_1, e_2 - e_1, -e_2\}$, the set of all transitions. If only the downstream server is uncontrolled, then its transition $D_2 x = x - e_2$, $x_2 > 0$ preserves submodularity and the system is transition-monotone. To see this, again note that submodularity of $f(D_2 x)$ at points on the boundary $x_2 = 0$ requires that f be convex in x_1 and increasing in x_1 and x_2 at these points. If, however, only the upstream server is uncontrolled, its transition $R_{12} x = x - e_1 + e_2$, $x_1 > 0$ does not preserve submodularity. Submodularity of $f(R_{12} x)$ at points with $x_1 = 0$ requires that f be increasing in the direction $e_1 - e_2$, which is not generally true of $V_n(x)$. Weber and Stidham [14] construct an example that is neither submodular nor transition-monotone.

Appendix

Proof of lemma

We need to show that, for $x \in X$ and $d_i \neq d_j \in \mathbf{D}(x)$,

$$\Delta = g(x + d_i) + g(x + d_j) - g(x) - g(x + d_i + d_j) \geqslant 0. \qquad \text{(A.1)}$$

Let $\mu(x)$ be the control that minimizes $g(x)$. Assume $\mu_k(x + d_i) \geqslant \mu_k(x + d_j)$.

Case I: $d_k \neq d_j$. If no boundaries are hit, submodularity of f and convexity of c_k give

$$\mu_k(x + d_i + d_j) \geqslant \mu_k(x + d_i) \geqslant \mu_k(x + d_j) \geqslant \mu_k(x),$$

or for brevity, $\mu_{i+j} \geqslant \mu_i \geqslant \mu_j \geqslant \mu$, motivating us to approximate μ_{i+j} by μ_i and μ by μ_j. Substituting into (A.1) and rearranging terms,

$$\begin{aligned}
\Delta \geqslant\ & c_k(\mu_i) + \mu_i f(x + d_i + d_k^1) + (\bar{\mu} - \mu_i) f(x + d_i + d_k^0) \\
& + c_k(\mu_j) + \mu_j f(x + d_j + d_k^1) + (\bar{\mu} - \mu_j) f(x + d_j + d_k^0) \\
& - c_k(\mu_j) - \mu_j f(x + d_k^1) - (\bar{\mu} - \mu_j) f(x + d_k^0) \\
& - c_k(\mu_i) - \mu_i f(x + d_i + d_j + d_k^1) - (\bar{\mu} - \mu_j) f(x + d_i + d_j + d_k^0) \\
=\ & (\bar{\mu} - \mu_j)\big[f(x + d_i + d_k^0) + f(x + d_j + d_k^0) - f(x + d_k^0) \\
& - f(x + d_i + d_j + d_k^0)\big] \\
& + (\mu_i - \mu_j)\big[f(x + d_i + d_j + d_k^0) + f(x + d_i + d_k^1) - f(x + d_i + d_k^0) \\
& - f(x + d_i + d_j + d_k^1)\big] \\
& + \mu_j\big[f(x + d_i + d_k^1) + f(x + d_j + d_k^1) - f(x + d_k^1) \\
& - f(x + d_i + d_j + d_k^1)\big].
\end{aligned}$$

By assumption (iii), all four points in the first bracketed expression are in X; hence, it is nonnegative by the submodularity of f w.r.t. d_i and d_j. For the second expression, $\mu_i > 0$ implies $x + d_i + d_k^1 \in X$. Also, $x + d_i + d_j + d_k^0 \in X$ by assumption (iii). Then, by compatibility of d_j and d_k, all four points are in X and the expression is nonnegative. Similarly, $\mu_j > 0$ implies $x + d_j + d_k^1 \in X$ and, since $\mu_i \geqslant \mu_j > 0$, $x + d_i + d_k^1 \in X$. Compatibility of d_i and d_j requires that all four points be in X, and the third expression is also nonnegative by the submodularity of f w.r.t. d_i and d_j.

Case II: $d_k = d_j$. Approximating μ_{i+j} by μ_j and μ by μ_i in (A.1) gives

$$\begin{aligned}
\Delta \geqslant\ & (\bar{\mu} - \mu_i)\big[f(x + d_i + d_k^0) + f(x + d_j + d_k^0) - f(x + d_k^0) \\
& - f(x + d_i + d_j + d_k^0)\big] \\
& + (\mu_i - \mu_j)\big[f(x + d_i + d_k^1) + f(x + d_j + d_k^0) - f(x + d_k^1) \\
& - f(x + d_i + d_j + d_k^0)\big] \\
& + \mu_j\big[f(x + d_i + d_k^1) + f(x + d_j + d_k^1) - f(x + d_k^1) \\
& - f(x + d_i + d_j + d_k^1)\big].
\end{aligned}$$

The first and third terms are nonnegative as before; the second term is zero, since $d_k = d_j$. □

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