Reinventing Heron

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Heron's proof of his formula for the area of a triangle, given only the lengths of the sides, is one of the more celebrated accomplishments of ancient mathematics (notably in [**3**]). Given sides of lengths a, b, c, we calculate half the perimeter $s = \frac{a+b+c}{2}$ and get the area formula $K = \sqrt{s(s-a)(s-b)(s-c)}$. Interestingly, there is little known about the progenitor of such a famous formula beyond his work and the consensus that it was done in Alexandria in the first century of the common era.

This elegant formula seems to be the kind of result that has to be rediscovered. The Indian mathematician Brahmagupta proves his "cyclic quadrilateral" formula (a generalization) in the seventh century, while the *Mathematical Treatise in Nine Sections* of Qin Jiushao (see [**7**]) shows an equivalent formula was known in China in the mid-13th century; apparently the Arabic mathematician al-Karkhi/Karaji also had a version which might or might not have been influenced by knowledge of Greek texts.

With this in mind, we will try to approach the problem of triangle area afresh, from a perhaps more modern viewpoint, focusing on *approximation* as a way forward. The genesis for this paper was when the geometry teacher of the second author's daughter took medical leave. To give her something to think about, he asked her about the area of a triangle; having forgotten Heron's formula, the stage was set for this line of investigation.

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Simple approximations

Let a, b, c be the lengths of the sides, with opposite vertices A, B, C. Our approach will be to start with two cases someone who has forgotten all their trigonometry, but still remembers the Pythagorean Theorem, can still analyze—right and isosceles triangles.

If we know any of the three altitudes, the usual formula "one half base times height" can be used. This is certainly the case for a right triangle. Assuming $a^2 + b^2 = c^2$, with b at the "base," then there is a right angle at C and the area is $K = \frac{1}{2}ab$.

Moreover, let's consider the set of *all* triangles (or, if you wish, representatives of isometry classes with congruent base line segment) with a given a and b. The one with a right angle at C has the greatest altitude (relative to the base b), so we obtain the simple approximation

$$K \leq \frac{1}{2}ab. \tag{1}$$

Next, consider an isosceles triangle with equal legs of length x, still considering b to be a 'base' (Figure 1). Its altitude is easily computed (by splitting the base and using Pythagoras) to be $h = \sqrt{x^2 - \left(\frac{b}{2}\right)^2}$, so in that case, $K = \sqrt{x^2\left(\frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^4}$.

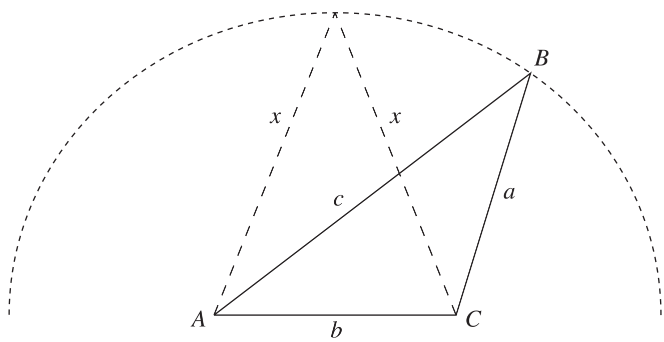

Figure 1. Isosceles approximation.

Now consider *any* triangle with a given b and $a + c$. For these triangles, vertex B lies on the ellipse with foci at A and C. The isosceles triangle has the greatest altitude of any such triangle, and equal legs $x = \frac{a+c}{2}$, so another bound is

$$K \leq \sqrt{\left(\frac{a+c}{2}\right)^2\left(\frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^4}. \tag{2}$$

If Heron was looking over our shoulder and had a bit of modern algebraic notation, he might be tempted to say something at this point. Notice the appearance of an average of some of the sides and a fourth power under the radical.

Approximations by perturbation

The pictures alone make it clear these bounds are accurate for triangles that are close to right or isosceles. The problem is that they can also be very *in*accurate; given a and

b, we can create a triangle with an arbitrarily small area with a very acute or obtuse angle $\angle CAB$.

We can hope to improve upon these approximations using calculus—and in particular, perturbation analysis. After all, the linear approximation should be more accurate than a constant, and higher-order approximations should be even more accurate.

Perturbing a circle. It will be simplest to analyze perturbations of a right triangle (Figure 2) leaving a and b fixed, as mentioned above. We will interpret x as the change in b due to perturbing the right triangle. (In the case when triangle ABC is acute, $x < 0$.)

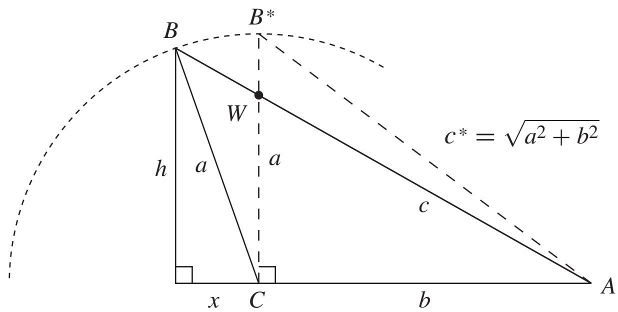

Figure 2. Circle perturbation.

For all such triangles, vertex B lies on a semicircle with center C. Notice that if we start with a right triangle and move vertex B, then the initial direction of motion is parallel to AC. In situations like this, the first order, or first derivative, effect is zero. Thus, a bound based on a first-order approximation is just the constant approximation given in (1). A second-order approximation in terms of the third length, c, can be constructed from the Taylor series

$$K = \frac{1}{2}ab + (c - c^*)\left.\frac{dK}{dc}\right|_{c=c^*} + \frac{1}{2}(c - c^*)^2 \left.\frac{d^2K}{dc^2}\right|_{c=c^*} + \cdots$$

where c^* is the hypotenuse of a *right* triangle (AB^*C in Figure 2) with legs a and b. As already noted, the first derivative is zero, since (letting h and x be as in Figure 2)

$$\frac{dK}{dc} = \frac{dK}{dh}\frac{dh}{dc} = \frac{b}{2}\frac{dh}{dc}$$

and $dh/dc|_{c=c^*} = 0$.

Analyzing the next term is trickier. We begin by noting that

$$\frac{d^2K}{dc^2} = \frac{b}{2}\frac{d^2h}{dc^2}.$$

Now, it is convenient to change variables from c to x (as indicated in Figure 2) in this derivative. In a very Leibnizian way, for small perturbations, we can assume AB is parallel to AB^* and the arc BB^* can be approximated by the (horizontal) tangent line at

B^*. Letting the intersection of BA and B^*C be W, we see that the triangles AB^*C and B^*WB are similar, so writing $c = c^* + \Delta c$, we have $x = (b/c)\Delta c + o(\Delta c)$. Thus,

$$\left.\frac{d^2h}{dc^2}\right|_{c=c^*} = \left(\frac{b}{c}\right)^2 \left.\frac{d^2h}{dx^2}\right|_{x=0}.$$

Now, the curvature of a circle is the reciprocal of the radius. The (signed) curvature of a function at a point where the slope is zero is also the second derivative. So $d^2h/dx^2|_{x=0} = -1/a$. Combining these three equations,

$$\left.\frac{d^2K}{dc^2}\right|_{c=c^*} = \frac{b}{2}\left(\frac{b}{c}\right)^2\left(-\frac{1}{a}\right) = -\frac{b^3}{2ac^2},$$

and a second-order approximation is

$$K = \frac{1}{2}ab - (c-c^*)^2\frac{b^3}{4ac^2} + o(\Delta c^2). \tag{3}$$

A parabolic fit. We obtained (3) without using the Pythagorean formula. Of course, to use (3), we must know c^*—if not from Pythagoras, then perhaps by measurement or by previous knowledge of (for example) a 3, 4, 5 triangle. To make more progress, let's explicitly use Pythagoras and a different Taylor polynomial. This will yield a simpler approximation.

Refer to Figure 2, in particular to the right triangle with solid borders. In this triangle, $(x+b)^2 + a^2 - x^2 = (x+b)^2 + h^2 = c^2$, so

$$x = \frac{c^2 - b^2 - a^2}{2b}. \tag{4}$$

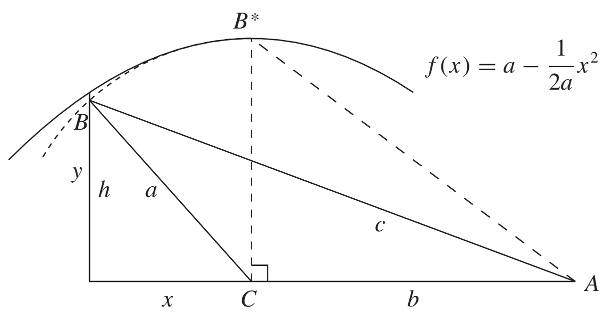

Figure 3. Parabolic perturbation.

Figure 3 shows a parabola with vertex B^* that fits the circle (dotted) in terms of slope and curvature—in other words, a second-degree Taylor polynomial. Using coordinate axes with the origin at C and x-axis CA, the equation of the parabola is $y = a - \frac{1}{2a}x^2$. The parabola lies above the circle, so $y \geq h$ and

$$K \leq \frac{1}{2}by = \frac{1}{2}b\left(a - \frac{1}{2a}x^2\right) = \frac{1}{2}ab - \frac{(c^2-a^2-b^2)^2}{16ab}. \tag{5}$$

This approximation has the same flavor as the one derived from the circle, but seems a bit simpler, and appropriately compares areas, not lengths, in the "correction" term from $\frac{1}{2}ab$. It uses geometry to find x, but still approximates y by a second-degree term using calculus. We are only one step away from finding the exact area—to which we will soon return.

Comparing accuracy. So how well do these approximations do? Given a sample triangle with $a = 17$ and $b = 25$, in Figure 4 we show the accuracy of the various approximations for some c near the right triangle case ($c^* = \sqrt{914} \approx 30.2$) and the isosceles case ($c = 25$). The isosceles bound uses $a = 25$, $b = 17$, which makes more sense for this range of c.

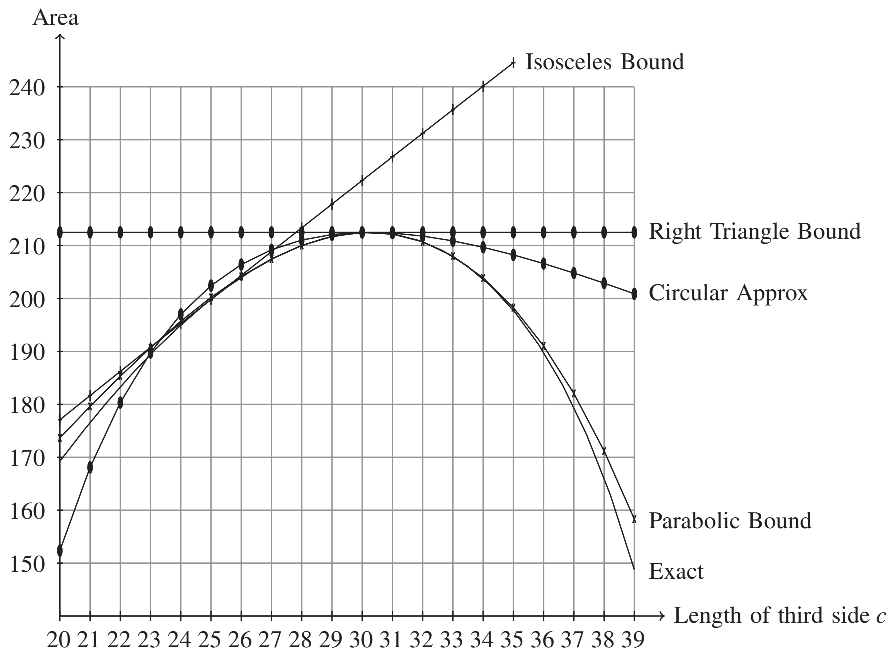

Figure 4. Comparing different approximations with $a = 17$ and $b = 25$.

Unsurprisingly, the isosceles approximation is exact for the isosceles case, but degrades just as quickly as the most naïve right case would, since it is a linear approximation. The second-order perturbation via the circle is accurate when reasonably close to a right triangle. The accuracy of the parabolic approximation (5) over a wide range of angles is impressive.

Figure 5 is similar, but with $a = 13$ and $b = 14$. We choose this because in the case $c = 15$, Heron's Formula gives exactly $\sqrt{21(8)(7)(6)} = \sqrt{7056} = 84$. Both approximations are extremely close, within 0.3% for this particularly nice triangle.

Back to Heron

We now return to the original question—can we come up with a formula for the area in terms of the three side lengths? Using the Pythagorean theorem and recalling (4),

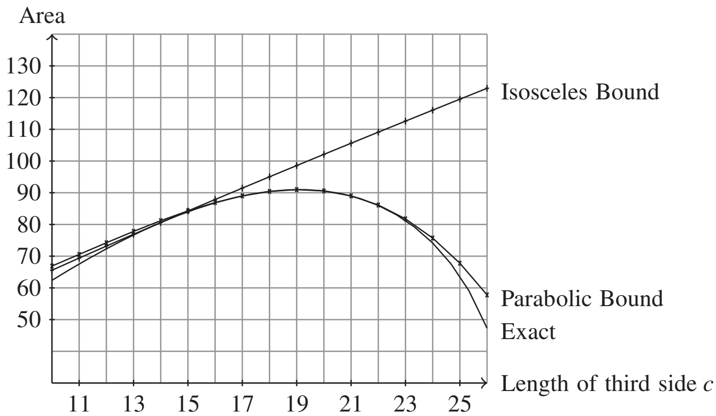

Figure 5. Comparing different approximations with $a = 13$ and $b = 14$.

we obtain

$$h = \sqrt{a^2 - x^2} = \sqrt{a^2 - \left(\frac{c^2 - a^2 - b^2}{2b}\right)^2},$$

which yields

$$K = \frac{1}{2}bh = \frac{1}{2}\sqrt{a^2b^2 - \frac{(c^2 - a^2 - b^2)^2}{4}}. \tag{6}$$

What is neat about equation (6) is that it gives the area of a triangle in terms of how far it is from a right triangle, as measured by $c^2 - a^2 - b^2$ (or $-2ab\cos\theta$, by the law of cosines); once again, the perturbation/approximation idea is of prime importance. It very clearly reduces to $\frac{1}{2}ab$ in the case of a right triangle. To see the relation with the isosceles bound (2), rewrite Heron's formula as

$$\begin{aligned} K &= \sqrt{\left(\frac{a+c}{2} + \frac{b}{2}\right)\left(\frac{b}{2} + \frac{a-c}{2}\right)\left(\frac{a+c}{2} - \frac{b}{2}\right)\left(\frac{b}{2} - \frac{a-c}{2}\right)} \\ &= \sqrt{\left[\left(\frac{a+c}{2}\right)^2 - \left(\frac{b}{2}\right)^2\right]\left[\left(\frac{b}{2}\right)^2 - \left(\frac{a-c}{2}\right)^2\right]}. \end{aligned}$$

This formula differs from (2) only through $a - c$, which may be viewed as the correction for the triangle being asymmetric. If $a = c$, the triangle is isosceles and both Heron's formula and (2) reduce to the formula for an isosceles triangle.

Conclusion

Heron's formula has had a remarkable life. Trying to approach it as something unknown has a certain intrigue; we particularly want to point out [**1**] as another example. Many other contemporary references may be found in the delightful [**6**], which is one of many resources showing not only that the Pythagorean theorem may be used to prove Heron's formula, but also that they are logically equivalent.

Of the many internet resources devoted to exploring it, we mention Cut-The-Knot [**2**] and Khan Academy [**5**]. At a different end of the historical timeline, the "perturbation" version of the area formula (6) is essentially the version Qin (see [**7**]) used centuries ago. In Problem III, 2, he uses as an example the 13, 14, 15 triangle we analyzed above.

We hope this paper stimulates the reader to see Heron's formula as more than a piece from the museum of mathematics, as beautiful as those pieces are. We think it is equally important to show how to come up with such a formula—or insights and approximations—from whatever tools one has at hand.

With that in mind, we offer some suggestions for exploring this approach in the classroom. We might ask how badly behaved these approximations are, along the lines of [**4**], which uses Heron's formula as a case study for numerical instability (with extremely acute triangles). Can we prove a tight bound on the parabolic approximation? When is it usable? What about approximations to Brahmagupta's cyclic quadrilateral formula?

Summary. Heron's formula gives the area of a triangle from the side lengths alone. Normally, this theorem is motivated by geometry or trigonometry. But how might calculus-style approximation lead us to such a formula? Here, we examine this idea, as well as the potential accuracy of such approximations.

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