Production Control with Backlog-Dependent Demand

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August 19, 2004

Abstract

We study a manufacturing firm that builds a product to stock to meet a random demand. Production time is deterministic, so that if there is a backlog, customers are quoted a lead time that is proportional to the backlog. In order to represent the customers’ response to waiting, we introduce a new defection function — the probability that a customer chooses not to order as a function of the backlog. Unlike models with backorder costs, the defection function is related to customer behavior. Using a continuous flow control model with linear holding cost and Markov modulated demand, we show that the optimal production policy has a hedging point form. The performance of the system under this policy is evaluated, allowing the optimal hedging point to be found.

1 Introduction

The purpose of this study is to modify the one-part-type, one-machine control problem of Bielecki and Kumar (1988a) to directly incorporate customer impatience. Bielecki and Kumar consider the problem in which a factory manager has to decide how to operate an unreliable machine to best satisfy a constant demand. Our model differs in two important respects.
1. **Backlog and customer impatience** In Kimemia and Gershwin (1983b) and Bielecki and Kumar (1988a), the difference between cumulative production and cumulative demand is called the *surplus*, represented by $x$. When $x$ is negative, it is *backlog*. The cost to be minimized is a function of $x$, which increases as $x$ deviates from 0. In this way, the optimization tends to keep $x$ near 0.

This makes economic sense for $x > 0$. In that case, $x$ is *finished goods inventory*, and there are clear, tangible costs associated with inventory (including the interest cost on the raw material, the floor space devoted to storage, etc.). However, there is often no such tangible cost associated with backlog (unless, for example, there is a contract with an explicit provision for a penalty for delivery delays). The undesirable consequence of backlog is the loss of sales, and lost sales are not related to backlog by a simple quantitative relationship.

One alternative to including an explicit cost term for $x < 0$ is to assume that there is no backlogging: When the surplus is zero, potential customers whose orders cannot be filled immediately leave and these lost sales reduce revenue. However, in many markets this does not adequately describe customer behavior.

We treat backlog in a more general way than either complete backlogging or no backlogging. In our model, if there is a positive surplus of finished goods, the customers make their purchases without delay and leave. If there is a backlog, some fraction of the customers are willing to wait to make their purchases, but others depart in disgust. Backlog is proportional to leadtime, so customers are reacting to a known leadtime. The greater the leadtime, the more customers leave without making a purchase. Some empirical studies demonstrating this behavior are discussed in the next section. Since accurate leadtimes are given upon arrival, we assume that once the customer is in the queue she does not renege (withdraw her order). In this way, we replace an artificial contrived cost term with a more natural model of the phenomenon that causes the cost.

2. **Reliable supply and variable demand** We assume a perfectly reliable factory in which leadtime is deterministic and proportional to backlog. Randomness in the model comes from the demand, which is assumed to be Markov modulated with two levels. Not only is random demand more realistic, but it also is more convenient to analyze in our setting than machine failures.

We only consider the effect of leadtime on present sales. Leadtime might have other consequences which we do not consider, such as a customer who finds the leadtime too great being less likely to attempt to make a purchase in the future, and the damage to a firm’s reputation when it has frequent large leadtimes. We do not consider the effect of price on customer behavior. Price is certainly important since reducing the price for long leadtime deliveries can persuade some customers not to defect. Finally, we do not consider the service or prices offered by competitors.

The objective is to maximize long-run average profit, where revenue is diminished by customers who are not willing to wait for their products and the cost rate is linear in the surplus. Unlike Bielecki and Kumar (1988a), there is no cost for backlog. We introduce a *defection function* $B(x)$ which represents the probability that a potential customer will not complete his order when the
backlog is \( x \). Then the instantaneous demand is reduced by a factor of \( 1 - B(x) \). \( B(x) \) describes the delay tolerance or impatience of the customer population. We find that the structure of the solution is the hedging point policy of Kimemia and Gershwin (1983b) and Bielecki and Kumar (1988a). The manufacturing facility produces with the maximum production rate until the hedging point is reached. When \( x \) is at the hedging level, the production rate is set to the demand rate and the surplus remains constant. We assume that in the high demand state demand exceeds capacity; consequently, the maximum production rate is always used (except in transient states).

When \( B(x) \) is piecewise constant, we solve analytically for the density function of the surplus under a hedging point policy. Since we can approximate any \( B(x) \) with a piecewise constant function, we can therefore solve systems with essentially any \( B(x) \). We express average profit as a function of the hedging point. Finally, we find the optimal hedging point numerically. To illustrate the behavior of the model, one family of defection functions is studied. Results are sensitive to both the median delay tolerance and the variability of the delay tolerance.

After a review of related work in Section 2, the model description and its assumptions are given in Section 3. In this section, backlog-dependent demand is introduced and the production control problem is stated. The optimal policy that maximizes the profit is characterized in Section 4. The model is analyzed and the steady state probability distributions are formulated in Section 5. Section 6 describes the evaluations of the objective function and of other performance measures of interest. The behavior of the model is investigated in Section 7. Section 8 contains a summary of the paper and several proposed research directions.

2 Related Work

2.1 Continuous control formulations of factory scheduling

Since the 1980s, there has been an increasing interest in devising optimal production control policies that manage production in uncertain environments. An optimal flow-rate control problem for a failure prone machine subject to a constant demand source was introduced by Olsder and Suri (1980) and Kimemia and Gershwin (1983b). The single-part-type, single-machine problem was analyzed in detail by Bielecki and Kumar (1988a). Hu (1995) extended this work to the case where no backlog is allowed.

Hedging point control policies are optimal or near optimal for a range of manufacturing system models. Hedging policies have been shown to be effective in a manufacturing environment by Yan, Lou, Sethi, Gardel, and Deosthali (1996). For a comprehensive survey, see Gershwin (1994).

Most of these studies assume a constant demand source. Only a few consider optimal production control problems with random demand, including Fleming, Sethi, and Soner (1987), Ghosh, Arapostathis, and Markus (1993), Tan (2002), and Perkins and Srikant (2001).

2.2 Queues with impatient customers

Impatience in queues includes reneging (abandoning the queue after waiting some time) and balking (not joining the queue if the server is not immediately available) (Hall 1991). Our model has queue-
length-dependent balking and no reneging. The implications of customer impatience have been addressed in a number of industries.

In a retail queueing model (Ittig 1994), the effect of waiting time on customer demand is taken into consideration when the optimal number of clerks for the queueing system is determined.

The behavior of internet users who react to the waiting times on the web is analogous to the behavior of customers who react to the waiting times in a manufacturing environment. Dellaert and Kahn (1999) report that the waiting times to load a web page can affect evaluation of web sites. When users experience long waits for a web site's home page to load, they either quit using the web or redirect to an alternative web page (Weinberg 2000).

Communicating waiting time information to customers is one way to improve the customer experience, according to Taylor (1994) and Hui and Tse (1996). In an $M/M/s/r$ queueing model with balking and reneging, Whitt (1999) shows that informing customers about anticipated delays can improve system performance.

In inventory management, most of the models assume that shortages are either completely lost or completely backlogged. In a recent study, partial backlogging and service-dependent sales are incorporated in a supply chain configuration study at Caterpillar Inc. (Rao, Scheller-Wolf, and Tayur 2000). Chang and Dye (1999) extended the basic economic order quantity model to include partial backlogging, where the backlog rate is inversely proportional to the waiting time for the next replenishment.

### 2.3 Empirical work on customer behavior

Fitzsimons (2000) investigates consumer responses to stockouts in terms of current customer satisfaction and in terms of subsequent store choice behavior. In this study, it is observed that the effect of a stockout depends on the customer's personal commitment to the out-of-stock item and also on the difficulty of making a choice from the other available items. A number of studies have investigated the effect of waiting time on customer demand in health care. In these studies, the effect is summarized by elasticity of demand with respect to waiting time. The effect of waiting time and private insurance premiums on the demand for public and private health care providers are estimated in the UK (McAvinchey and Yannopoulos 1993). In another study that investigates the rationing effect of waiting, elasticity of demand with respect to waiting time is estimated empirically using the waiting list for elective surgery in the British National Health Service (Martin and Smith 1999). Another study investigated service time competition between companies by using two identical gasoline service stations (Mount 1994). In a recent study, Anderson, Fitzsimons, and Simester (2003) investigate the effects of stockouts on profits by using three years of purchasing behavior for 20,000 customers of a mail-order catalog. In this study, it is also observed that, when an item is out of stock, the percentage of customers who cancel their orders increases with the anticipated delay before the item is expected to ship.
3 Model Description

In this section, we describe the problem and the model assumptions. The structure of the solution is described in Section 4.

3.1 Basic Model

We consider a make-to-stock system with a single manufacturing facility that produces to meet the demand for a single item. Production, demand, inventory, and backlog are all represented by continuous (real) variables. The demand rate at time $t$ is denoted by $d(t)$. The state of the demand at time $t$ is $D(t)$ which is either high (H) or low (L). When the demand is high, the demand rate is $d(t) = \mu_H$ and when the demand is low, the demand rate is $d(t) = \mu_L$. At time $t$, the amount of finished goods inventory or backlog is $x(t)$.

The times to switch from a high demand state to a low demand state and from a low demand state to a high demand state are assumed to be exponentially distributed random variables with rates $\lambda_{HL}$ and $\lambda_{LH}$. This model is suitable for describing demand which is stationary in the long run, but whose mean shifts temporarily as a result of promotions, competitor actions, etc. The time since the last state change does not change the expected time until the next state change. The average demand rate is

$$Ed = \mu_H e + \mu_L (1 - e)$$

where $e = \lambda_{LH}/(\lambda_{HL} + \lambda_{LH})$ is the percentage of the time the demand is high.

The limiting variance rate of the amount demanded during a period of length $t$ satisfies

$$Vd = \lim_{t \to \infty} \frac{V[N(t)]}{t} = \frac{2(\mu_H - \mu_L)^2 (1 - e) e^2}{\lambda_{LH}}$$

and the total amount demanded during a period of length $t$ is asymptotically normal as $t \to \infty$ (Tan 1997).

The asymptotic coefficient of variation of the demand, defined as $cv = \sqrt{Vd}/Ed$, is used as a summary measure of the demand variability in the numerical experiments.

The maximum production rate of the manufacturing facility is $u$. The actual production rate of the manufacturing facility at time $t$ is a control variable which is denoted by $u(t)$, $0 \leq u(t) \leq u$. We assume that the production capacity $u$ is sufficient to meet the demand when it is low but insufficient when it is high, i.e., $\mu_L < u < \mu_H$. (Note that if $u > \mu_H$, the problem is trivial: it is always possible to keep $x$ at 0 and therefore a backlog situation never happens. Similarly, if $u < \mu_L$, the problem is also trivial: the manufacturing facility is run at the maximum rate all the time.)

The reward per unit for goods produced in the factory is $A$ and the inventory carrying cost is $g^+$ (dollars per unit per time). As indicated earlier, we do not include the corresponding backlog cost $g^-$, which does appear in Bielecki and Kumar (1988a) and many other papers.
3.2 Backlog-Dependent Demand

When there is backlog (i.e., when \( x < 0 \)), a potential customer chooses not to order with probability \( B(x, D) \) when the backlog is \( x \) and the demand state is \( D \). (Alternatively, \( B(x, D) \) is the fraction of potential customers who choose not to order when the backlog is \( x \) and the demand is \( D \).) \( B(x, D) \), which is called the \textit{defection function}, satisfies

\[
\begin{align*}
0 \leq B(x, D) &\leq 1, \\
x > 0 &\implies B(x, D) = 0, \\
B(0, L) & = 0, \\
B(0, H) & = \min \left\{ \frac{\mu_H - u}{\mu_H}, B(0^-, H) \right\}, \\
\text{For some } x, B(x, D) & \geq \frac{\mu_H - u}{\mu_H}.
\end{align*}
\] (1)

The first condition is required by the definition of \( B \) as a probability or a fraction. The second says that no potential customers are motivated to defect when there is surplus.

The third and fourth condition prevent excess defection when \( x = 0 \). Only customers with no patience and who cannot be served immediately should defect. Thus, when \( D = L \) no customers defect and when \( D = H \) the fraction defecting is the minimum of the fraction with no patience and the fraction that could not be served immediately. Note that we could have set \( B(0) = 0 \), avoiding the dependence of \( B \) on \( D \), but this would make chattering possible in state \((0, H)\). Since the dependence on \( D \) is merely a technical convenience, in the remainder of the paper we suppress the dependence on \( D \) in our notation, and write \( B(x) \).

The last condition guarantees that \( x(t) \) is bounded from below. It is implied by the natural condition \( \lim_{t \to \infty} B(x) = 1 \), which says that nobody is infinitely patient. This condition is further discussed at the end of Section 3.3.

If \( B(x) \) satisfies

\[ B(x) \text{ is a non-increasing function of } x, \]

we say that \( B \) is a \textit{monotonic defection function}. If \( B \) is monotonic, more customers are impatient if there is a longer wait. In the following, we restrict our attention to monotonic defection functions. Lastly, we assume that \( B(x) \) is right continuous left limit for \( x < 0 \). Figure 1 shows an example of a \( B(x) \) function that satisfies these conditions.

In the Appendix of Tan and Gershwin (2001), the customer defection function that is generated by customers who choose the shortest of two queues is derived. This preliminary analysis yields a \( B(x) \) function similar to the one depicted in Figure 1. The empirical study of Anderson, Fitzsimons, and Simester (2003) also support the form of the defection function suggested in this study. Analysis
of alternative functional forms of $B(x)$ resulting from different customer behavior is left for future research.

Note that $B(x) = 1$ for all $x < 0$ corresponds to the lost sales case. In this situation, no customers are willing to wait to receive their goods. All revenues are lost whenever there is any backlog.

When the surplus level is $x < 0$, the time until the next arriving customer order will be completed, i.e., the production lead time, is $-x/u$. This is the time to clear the current backlog, assuming $u = u$ until the backlog is cleared. The leadtime dependent demand case can therefore be treated by using $B(x) = B(-ut_q) = \tilde{B}(t_q)$ as the probability that a potential customer chooses not to order when the quoted lead time is $t_q = -x/u$.

### 3.3 Production Control Problem

The decision variable is the rate at which the goods are produced at the plant at time $t$, $u(t)$. The dynamics of $x$ are given by

$$\frac{dx}{dt} = u(t) - d(t)(1 - B(x))$$

at points where the derivative exists.

In earlier versions of the production control problem (for example Kimemia and Gershwin 1983a; Bielecki and Kumar 1988b), the $(1 - B(x))$ factor did not appear. (That is, $B$ was always 0.) It
is present in this version to represent the lost demand due to customer balking. The quantity $d(t)(1 - B(x))$ is the rate at which sales actually take place, and is therefore the rate at which the surplus/backlog is diminished by sales.

The revenue rate at time $t$ is therefore $Ad(t)(1 - B(x(t)))$ The profit rate is the difference between the revenue generated through demand and the inventory carrying costs, which are assumed to be linear:

$$g(x, D) = Ad(t)(1 - B(x)) - g^+x^+$$

The profit of policy $u(t)$ in the interval $[0, T]$, $J^u(x, D; T)$, is defined as

$$J^u(x, D; T) = E_{x,D} \int_0^T g(x(t), D(t))dt,$$

where $E_{x,D}$ denotes expectation with respect to the initial state $x(0) = x$, $D(0) = D$. Similarly, the long-run average profit of policy $u$ is defined as

$$\overline{J}^u = \limsup_{T \to \infty} \frac{1}{T} J^u(x, D; T).$$

The production control problem is

$$\mathcal{J} = \max_u J^u$$

subject to

$$\frac{dx}{dt} = u(t) - d(t)(1 - B(x))$$

$$0 \leq u(t) \leq u$$

$$d(t) = \begin{cases} \mu_H & \text{if } D = H \\ \mu_L & \text{if } D = L \end{cases}$$

Markov dynamics for $D$ with rates $\lambda_{HL}$ from $H$ to $L$ and $\lambda_{LH}$ from $L$ to $H$ (7)

We do not include an explicit cost for backlog in the definition of $g$. Instead, the firm is penalized for backlog by the defection of impatient customers.

The system is stable under some policy if $Ed(1 - \lim_{t \to \infty} B(x)) < u$ or $\lim_{t \to \infty} B(x) > \frac{(Ed - u)}{Ed}$. In fact, there is no reason for $u$ to be anything but maximal when $x$ is negative. Then (1) implies that $x$ is bounded from below. To see this, let $x^* = \max\{x : u - \mu_H(1 - B(x)) \geq 0\}$ and assume that $x(0) > x^*$. From (4), $dx/dt \geq 0$ for all $x \leq x^*$, for $D = L$ or $H$ (since $\mu_H > \mu_L$). Then $x(t) \geq x^*$ for all $t$. Even if $Ed$ is greater than $u$, enough impatient customers will defect to guarantee that $x$ is bounded from below.
4 Characterization of the Policy

The solution of problem (3)–(7) is a stationary feedback control \( u = u(x, D) \) that satisfies the Bellman equation. Define the differential cost (actually reward) for policy \( u \) as

\[
V^u(x, D) = \lim_{T \to \infty} J^u(x, D; T) - T \mathcal{T}^u
\]

and the differential cost for the optimal policy by \( V \) when these limits exist. \( V \) satisfies the maximum principle, which asserts that, assuming that \( \partial V/\partial x \) exists,

\[
\mathcal{T} = \max_u \left\{ -g(x, L) + \frac{\partial V}{\partial x}(x, L)(u - \mu_L(1 - B(x))) + [V(x, H) - V(x, L)]\lambda_{HL} \right\} \tag{8}
\]

for \( D = L \), and

\[
\mathcal{T} = \max_u \left\{ -g(x, H) + \frac{\partial V}{\partial x}(x, H)(u - \mu_H(1 - B(x))) + [V(x, L) - V(x, H)]\lambda_{LH} \right\} \tag{9}
\]

for \( D = H \). The maximizations are taken over constraints (5).

The differentiability of \( V \) in intervals of constant control follows from the general theory of jump Markov processes in Rishel (1975). Differentiability where the control changes could be proven using the methods of Sethi, Suo, Taksar, and Zhang (1997). However, we can avoid this issue by replacing \( \partial V/\partial x \) by the appropriate one-sided derivative, namely, \( \partial^+ V/\partial x \) when \( \frac{dx}{dt}(t^+) \geq 0 \) and \( \partial^- V/\partial x \) when \( \frac{dx}{dt}(t^+) < 0 \). The existence of one-sided derivatives follows from Rishel’s theory.

The optimal policy has a very simple form.

**Theorem 1** The optimal policy has hedging point form: For some \( Z(L) \geq 0 \),

\[
u^*(x, D) = \begin{cases} \mu_L & \text{if } x = Z(L) \text{ and } D = L \\ u & \text{if } x < Z(L) \end{cases}
\]

A proof is given in the Appendix, based on comparing the problem to the lost sales unreliable machine problem. The policy on the transient states \( x > Z(L) \) does not affect the average cost. However, it seems apparent from similar problems that the optimal policy on these states uses thresholds \( 0 \leq Z(L) \leq Z(H) \), and that the complete policy is

\[
u^*(x, D) = \begin{cases} 0 & \text{if } x > Z(L) \text{ and } D = L \\ \mu_L & \text{if } x = Z(L) \text{ and } D = L \\ u & \text{if } x < Z(L) \text{ and } D = L \\ 0 & \text{if } x > Z(H) \text{ and } D = H \\ u & \text{if } x \leq Z(H) \text{ and } D = H \end{cases}
\]

Note that \( Z(H) \) is not a hedging point; \( dx/dt < 0 \) at \( x = Z(H), D = H \).
5 Analysis of the Model

In this section, we analyze the model with backlog-dependent demand. We calculate the steady-state probability distribution of $x$ and $D$ assuming that the system is operated under the policy of Section 4. In Section 6, we evaluate the expected profit (as well as other performance measures). Then we find the optimal policy by finding the value of $Z(L)$ that maximizes the expected profit.

5.1 Dynamics

The analysis of even simple systems with general non-zero $B(x)$ results in non-closed form solutions. In order to treat a wide variety of backlog-dependent demand functions, it is convenient to assume that $B(x)$ is piecewise constant. That is,

$$B(x) = \begin{cases} 0 & x > 0 \\ B_1 & 0 \geq x > \beta_1 \\ B_i & \beta_{i-1} \geq x > \beta_i & i = 2, \ldots, M \end{cases}$$

(10)

where $\beta_i$, $B_i$, $i = 1, \ldots, M$ are constants, $0 > \beta_i > \beta_{i+1}$. From (1) and (2),

$$0 \leq B_i < B_{i+1} \leq 1,$$

By a proper choice of these constants, and for large enough $M$, any monotonic $B(x)$ can be arbitrarily closely approximated. Figure 2 shows a step-wise constant approximation of a continuous $B(x)$ function.

It is also convenient to include the lower bound on $x$ in the discretization of $B(x)$. In order to analyze only the recurrent states, $B_M$ can be chosen such a way that it satisfies

$$u = \mu_H(1 - B_M)$$

Let us define region boundaries $R_0 > R_1 > \ldots > R_J$ with

- for $Z(L) > 0$: $R_0 = Z(L)$, $R_1 = 0$, $R_i = \beta_{i-1}$ for $i = 2, \ldots, M + 1$ and $J = M + 1$,
- for $Z(L) = 0$: $R_0 = 0$, $R_i = \beta_i$ for $i = 1, \ldots, M$, and $J = M$.

The recurrent values of $x$ are $[R_J, R_0]$. Call $(R_{i+1}, R_i)$ region $i$. Within each of these regions, the right side of the $x$ dynamics is constant and $g(x, D)$ is linear.

Let $\Delta_i^L$ be the rate of change of $x$ in region $i$ when the demand state $D$ is low (L). Then

$$\Delta_i^L = u - \mu_L(1 - B(x)), \; R_i < x < R_{i+1}, \; i = 0, 1, \ldots, J - 1$$

(11)

where $u$ and $B(x)$ are constant in each region, so $\Delta_i^L$ is constant.

Similarly, let $\Delta_i^H$ be the rate of change of $x$ in region $i$ when $D = H$. Then

$$\Delta_i^H = u - \mu_H(1 - B(x)), \; R_i < x < R_{i+1}, \; i = 0, 1, \ldots, J - 1$$

(12)
Figure 2: Step-wise constant approximation of a continuous $B(x)$ function
A sample path of a system where $B(x)$ is given as a five level step-wise constant function is depicted in Figure 3. When the demand is high, $u = u_d$. However, since $u < \mu_H$, the surplus decreases with rate $\Delta^H_0 = u - \mu_H$ for $x > 0$ (Region 0). For $x < 0$, $B(x)$ increases step-by-step from $\beta_1$ to $\beta_5$. When $x = \beta_5$, the demand rate of the customers who have not defected is equal to the maximum production rate and therefore, the backlog stays at the lower bound until the demand switches to low. When the demand is low, $x$ increases with a step-wise constant slope until it reaches the hedging level at $Z(L)$.

In the following sections, we describe how the optimal policy is determined. First, the system is evaluated by determining the probability density functions in the interior, and probability masses at the upper and lower levels for a given value of $Z(L)$. Then, the optimal value of the hedging level is determined by maximizing the expected profit.

### 5.2 Probability distribution

When the surplus/backlog $x$ is not equal to the upper or lower levels ($R_0$ or $R_J$), the system is said to be in the *interior*. The system state at time $t$ is $S(t) = (x(t), D(t))$ where $R_J < x(t) < R_0$ and $D(t) \in \{H, L\}$. 
The time-dependent system state probability distribution for the interior region, $F_D(t, x)$, is defined as
\[
F_D(t, x) = \text{prob}[D(t) = D, x(t) \leq x], \quad t \geq 0, \quad D \in \{H, L\}, \quad R_J < x(t) < R_0
\] (13)

The time-dependent system state density functions are defined as
\[
f_D(t, x) = \frac{\partial F_D(t, x)}{\partial x} \quad t \geq 0, \quad D \in \{H, L\}, \quad R_J < x(t) < R_0
\] (14)

We assume that the process is ergodic and, thus, the steady-state density functions exist. The steady-state density functions are defined as:
\[
f_D(x) = \lim_{t \to \infty} f_D(t, x), \quad D \in \{H, L\}, \quad R_J < x(t) < R_0.
\] (15)

It is possible to show ergodicity by observing that in the Markov process model, all of the states constitute a single communicating class. It is also possible to demonstrate aperiodicity.

### 5.3 Region $i$: $R_{i+1} < x < R_i$

Suppose $R_{i+1} < x(t + \delta t) < R_i$, $i = 1, 2, \ldots, J - 1$, and $D(t + \delta t) = H$. Then, since we are modeling this system as a Markov process,
\[
f_H(t + \delta t, x) = f_H(t, x - \Delta_i^H \delta t)(1 - \lambda_{HL} \delta t) + f_L(t, x)(\lambda_{LH} \delta t) + o(\delta t)
\] (16)

where $o(\delta t)$ approaches to zero faster than $\delta t$. This equation can be written in differential form, for $\delta t \to 0$, as
\[
\frac{\partial f_H(t, x)}{\partial t} + \Delta_i^H \frac{\partial f_H(t, x)}{\partial x} = -\lambda_{HL} f_H(t, x) + \lambda_{LH} f_L(t, x)
\] (17)

Taking the limit of (17) as $t \to \infty$ yields the following steady-state differential equation for $f_H(x)$:
\[
\Delta_i^H \frac{df_H(x)}{dx} = -\lambda_{HL} f_H(x) + \lambda_{LH} f_L(x), \quad R_{i+1} < x < R_i
\] (18)

Following the same steps for $f_L$ yields
\[
\Delta_i^L \frac{df_L(x)}{dx} = \lambda_{HL} f_H(x) - \lambda_{LH} f_L(x) \quad R_{i+1} < x < R_i
\] (19)

In order to solve the set of first order differential equations given in (18) and (19), two boundary conditions for each region are needed. First, note that at any given level of the finished goods inventory, the number of upward crossings must be equal to the number of downward crossings. Let $N(D, \xi, T)$ denote the total number of level crossings in demand state $D$, at surplus level $\xi$, in the time interval $[t, t + T]$. Then
\[
\lim_{T \to \infty} N(H, \xi, T) = \lim_{T \to \infty} N(L, \xi, T)
\]  
(20)

Renewal analysis shows that
\[
\lim_{T \to \infty} \frac{N(D, \xi, T)}{T} = \Delta_i^P f_D(\xi)
\]  
(21)

where \(\Delta_i^P\) is the rate of change in the buffer level when the demand state is \(D\) and \(\xi\) is in region \(i\), and \(f_D(\xi)\) is the steady-state density function. This kind of analysis was also employed by Yeralan and Tan (1997). Then, equation (20) can be written as
\[
-\Delta_i^H f_H(x) = \Delta_i^L f_L(x).
\]  
(22)

Using this result in equation (18) gives the following first order differential equation
\[
\frac{df_H(x)}{dx} = \left( -\frac{\lambda_{HL}}{\Delta_i^H} - \frac{\lambda_{LH}}{\Delta_i^L} \right) f_H(x)
\]  
(23)

whose solution is
\[
f_H(x) = c_i e^{\eta_i x}, \quad R_{i+1} < x < R_i
\]  
(24)

where
\[
\eta_i = -\frac{\lambda_{HL}}{\Delta_i^H} - \frac{\lambda_{LH}}{\Delta_i^L}
\]
and \(c_i\) is a constant to be determined. Following equation (22),
\[
f_L(x) = -c_i \frac{\Delta_i^H}{\Delta_i^L} e^{\eta_i x}, \quad R_{i+1} < x < R_i
\]  
(25)

### 5.4 External Boundary Conditions

The steady-state probabilities \(P^0\) and \(P^J\) that the finished goods inventory is equal to the hedging level \(Z(L)\) and the lowest level \(X\) are defined as
\[
P^0 = \lim_{t \to \infty} \text{prob}\{x(t) = R_0\},
\]  
(26)
\[
P^J = \lim_{t \to \infty} \text{prob}\{x(t) = R_j\}.
\]  
(27)

The inventory level can increase only when the demand is low. When \(x\) increases and reaches the level \(R_0 = Z(L)\), the inventory level stays at the upper level until the demand rate increases to \(\mu_H\) and \(x\) starts decreasing. The expected remaining time for the state of the demand to change from \(L\) to \(H\) is \(1/\lambda_{LH}\). Then \(P^0\) is fraction of time that \(x = Z(L)\):
\[ P^0 = \lim_{T \to \infty} \frac{N(L, R_0, T)}{T} \frac{1}{\lambda_{LH}} \Delta^L_0 f_L(R_0) \frac{1}{\lambda_{LH}} = -c_0 \frac{\Delta^H_0}{\lambda_{LH}} e^{\eta R_0}. \] (28)

Similarly
\[ P^J = \lim_{T \to \infty} \frac{N(H, R^+_j, T)}{T} \frac{1}{\lambda_{HL}} = -\Delta^H_{j-1} f_H(R_j) \frac{1}{\lambda_{HL}} = -c_{j-1} \frac{\Delta^H_{j-1}}{\lambda_{HL}} e^{\eta_{j-1} R_j}. \] (29)

Let us also define \( P^H_i \) and \( P^L_i \) \( i = 0, 1, \ldots, J-1 \) as the probabilities that the process is in region \( i \) in the long run when the demand is high and when it is low, respectively:
\[ P^H_i = \lim_{t \to \infty} \text{prob}[R_i < x(t) < R_{i+1}, D(t) = H] \quad i = 0, \ldots, J-1 \] (30)
\[ P^L_i = \lim_{t \to \infty} \text{prob}[R_i < x(t) < R_{i+1}, D(t) = L] \quad i = 0, \ldots, J-1 \] (31)

Once the density functions are available, \( P^H_i \) and \( P^L_i \) can be evaluated as
\[ P^H_i = \int_{R_{i+1}}^{R_i} f_H(x) dx \quad i = 0, \ldots, J-1 \] (32)
\[ P^L_i = \int_{R_{i+1}}^{R_i} f_L(x) dx \quad i = 0, \ldots, J-1 \] (33)

### 5.5 Internal Boundary Conditions

To complete the derivation of the density functions, the coefficients \( c_i, i = 0, 1, \ldots, J-1 \) must be determined. Since there are \( J \) unknowns, \( J \) boundary conditions are needed. The \( J-1 \) internal boundary conditions come from the equality of the number of upward and downward crossings at the region boundaries. For large \( T \),
\[ \lim_{T \to \infty} N(j, R^+_i, T) = \lim_{T \to \infty} N(j, R^-_i, T), j \in \{H, L\}, i = 1, 2, \ldots, J-1. \] (34)

By using equation (21), this equation can be written
\[ \Delta^j_{i-1} f_j(R^+_i) = \Delta^j_{i-1} f_j(R^-_i), j \in \{H, L\}, i = 1, 2, \ldots, J-1. \] (35)

### 6 Solution of the Model

#### 6.1 Coefficients

Writing (35) in terms of the solution of the density function for \( j = H \) given in equation (24) yields
\[ \Delta^H_{i-1} c_{i-1} e^{\eta_{i-1} R_i} = \Delta^H_i c_i e^{\eta R_i}, i = 1, 2, \ldots, J-1, \] (36)
or

\[ c_i = \frac{\Delta H_i}{\Delta H_{i-1}} e^{(\eta_i - \eta_{i-1})R_i} c_{i-1}, \quad i = 1, 2, ..., J - 1 \]  

(37)

Then all the constants \( c_i \), \( i = 1, 2, ..., J - 1 \) can be determined by \( c_0 \), since

\[ \phi_i = \prod_{j=1}^{i-1} \frac{\Delta H_{j-1}}{\Delta H_j} e^{(\eta_j - \eta_j)R_j}, \quad i = 1, ..., J - 1 \]  

(38)

and \( \phi_0 = 1 \).

Finally, the constant \( c_0 \) is determined by using the normalizing condition. The sum of all the probabilities must add up to 1, or

\[ \sum_{i=0}^{J-1} (P_i^H + P_i^L) + P_J = 1 \]  

(39)

Equations (24), (25), (28), (29), (32) and (33) yield

\[ c_0 = \left[ \frac{(\mu_H - \mu)e^{\eta R_0}}{\lambda_H} + \sum_{i=0}^{J-1} \phi_i \frac{\Delta L_i - \Delta H_i}{\Delta L_i} X_i - \frac{\phi J \Delta H_{J-1} e^{\eta H_{J-1} R_J}}{\lambda_H} \right]^{-1} \]  

(40)

where

\[ X_i = \begin{cases} 
(\eta_i R_i - \eta_i R_{i+1})/\eta_i & \text{if } \eta_i \neq 0, \\
(R_i - R_{i+1}) & \text{if } \eta_i = 0.
\end{cases} \]

6.2 Evaluation of the Objective Function

In order to determine the optimal values of the hedging levels, the profit must be evaluated. Let \( \Pi \) be the average revenue rate generated by the plant. \( \Pi \) is given by

\[ \Pi = \lim_{T \to \infty} E \left[ \frac{1}{T} \int_0^T A d(\tau)(1 - B(x(\tau)) d\tau) \right] \]

From (4), this is

\[ \lim_{T \to \infty} E \left[ \frac{1}{T} \int_0^T A \left( u - \frac{dx}{dt} \right) d\tau \right] = \lim_{T \to \infty} E \left[ \frac{1}{T} \int_0^T A d\tau - \frac{A}{T} (x(T) - x(0)) \right] \]

But we know that \( E[x(T) - x(0)] \) will be bounded since the system is stable. Since the difference between the cumulative production and cumulative demand is finite in the long run, the profit term in the objective function can also be written using the production rate rather than the demand rate. Therefore,
\[ \Pi = \lim_{T \to \infty} E \left[ \frac{1}{T} \int_0^T Aud\tau \right] = A \left( \text{prob}[x < R_0]u + \text{prob}[x = R_0]u_L \right) \] (41)

which can be simplified as

\[ \Pi = A \left( u - c_0 \frac{(u - \mu_L)(\mu_H - u)}{\lambda_H} e^{\eta_0 R_0} \right) \] (42)

Profit also depends on the inventory carrying costs. Let \( \Psi_i \) be defined as

\[ \Psi_i = \int_{R_i}^{R_i+1} x (f_H(x) + f_L(x)) \, dx = c_i \frac{\Delta_i^L - \Delta_i^H}{\Delta_i^L} Q_i \] (43)

where

\[ Q_i = \left\{ \begin{array}{ll}
(\eta_i R_i - 1) e^{\eta_i R_i} - (\eta_i R_{i+1} - 1) e^{\eta_i R_{i+1}} / \eta_i^2 & \text{if } \eta_i \neq 0, \\
(R_i^2 - R_{i+1}^2) / 2 & \text{if } \eta_i = 0.
\end{array} \right. \]

The average inventory level \( E_{WIP} \) is

\[ E_{WIP} = \Psi_0 I_{\{Z(L)>0\}} + R_0 P^0 \] (44)

where \( I_{\{Z(L)>0\}} \) is an indicator function which is 1 if \( Z(L) > 0 \) and 0 otherwise.

Finally, the average profit per unit time in (3) is

\[ \overline{J} = \Pi - g^+ E_{WIP} \] (45)

The optimal values of \( Z(L) \) and \( Z(H) \) are determined by maximizing \( \overline{J} \).

### 6.3 Other Performance Measures

We can also evaluate other quantities of interest. The average sales rate or throughput rate is

\[ TH = \lim_{T \to \infty} E \left[ \frac{1}{T} \int_0^T u d\tau \right] = u - c_0 \frac{(u - \mu_L)(\mu_H - u)}{\lambda_H} e^{\eta_0 R_0} \] (46)

The service level, the ratio of the average sales to the average demand, is

\[ SL = \frac{TH}{Ed} \] (47)

The fill rate is the probability that a customer receives his product as soon as he arrives:

\[ FR = \text{prob}[x \geq 0] = P^0 + (P^H_0 + P^L_0) I_{\{Z(L)>0\}} \]

The average backlog level is \( E_{BL} \) which is evaluated as
\[ E_{BL} = -\Psi_0 I_{\{Z(L) = 0\}} - \sum_{j=1}^{J-1} \Psi_j - R_j P^j \] (48)

7 Behavior of the Model

7.1 Effect of Customer Defection Behavior

In the numerical examples, \( B(x) \) is a sigmoid function in the form

\[ B(x) = \frac{1}{1 + e^{\gamma(x-\eta)}}. \]

Figure 1 shows a sigmoid function with \( \gamma = 1/2 \) and \( \eta = -10 \).

In order to capture the customers’ sensitivity to the quoted lead time in a simple way, preferably with a single parameter, the maximum backlog after which almost all the customers choose not to order is referred as \( \chi \). More specifically, let \( \chi \) be the value of \( x \) that yields \( B(x) = 1 - \epsilon \) where \( \epsilon \) is a very small positive number. Then \( \eta \) is set to \( \frac{\chi}{2} \) and \( \gamma \) is determined by \( \chi \) as \( \gamma = -\frac{2}{\chi} \ln[\frac{1}{\epsilon} - 1] \).

Figure 4 shows various \( B(x) \) functions for different values of \( \chi \).

In the discretization of \( B(x) \), the step size is set to \( \delta = \frac{1}{M\gamma} \ln[\frac{1}{1-\epsilon} + \chi] \) in order to reach \( 1 - \epsilon \) in \( M \) steps. Then \( \beta_0 \) is set to 0.

\[
\begin{align*}
\beta_i &= -\delta i, \\
B_i &= \frac{1}{2} \left( \frac{1}{1 + e^{\gamma(\beta_{i-1} - \chi)}} + \frac{1}{1 + e^{\gamma(\beta_i - \chi)}} \right)
\end{align*}
\]

and, in these cases, \( \epsilon = 0.001 \) and \( M = 10 \).

The effect of customer defection behavior on the performance of the system is examined by considering the effects of the maximum delay tolerance of the customers and the way different customers defect for the same quoted lead time separately.

The effect of the customers’ sensitivity to delay on the performance of the system is depicted in Figure 5. As customers become more sensitive to backlog, i.e., as \( \chi \) increases, the profit and the service level decrease, and the expected inventory level increases. Due to the loss of more and more customers, expected backlog level also decreases and the service level is low. The upper and lower boundaries (\( Z_L \) and \( x^* \)) increase as the customers become more sensitive to the waiting time.

In order to investigate the effect of the form of the defection function with the same maximum delay tolerance, i.e., with the same \( \chi \), the steepness of \( B(x) \) is varied by changing \( \eta \) and setting \( \gamma = -\frac{1}{\chi-\eta} \ln[\frac{1}{\epsilon} - 1] \) to reach \( B(x) = 1 - \epsilon \) at \( x = \chi \). Figure 6 shows different \( B(x) \) functions for different values of \( \eta \) with \( \chi = -10 \).

Figure 7 depicts the effects of the steepness of the customer defection function on the performance of the system for the same maximum delay tolerance of \( \chi = -10 \). As \( B(x) \) becomes steeper, i.e. as \( \eta \) approaches \( \chi \), fewer customers defect until the maximum delay tolerance is reached. Therefore the optimal profit \( J \) and \( SL \) increases. Moreover, carrying less inventory at the expense of increased
Figure 4: Various $B(x)$ functions for different values of $\chi$
Figure 5: Effect of Maximum Delay Tolerance ($\mu_H = 1.5$, $\mu_L = 0.3$, $\lambda_{HL} = 0.05$, $\lambda_{LH} = 0.05$, $\mu = 0.6$, $A = 3$, $g^+ = 0.1$, $M = 10$)
Figure 6: Various $B(x)$ functions for different values of $\eta$ with $\chi = -10$
backlog becomes feasible. Similarly, the upper hedging level $Z(L)$ decreases and approaches to zero. The lower limit $x^*$ also decreases and approaches $\chi$. As a result, $E_{WIP}$ decreases while $E_{BL}$ increases.

### 7.2 Effect of Demand Variability

The effect of the variability of demand on the performance of the system is depicted in Figure 8. The variability of demand is indicated by the coefficient of variation of the demand rate $cv$ that is defined in Section 3.1. As the demand variability increases, the profit and service level decrease and the average backlog and the average inventory increase. Increasing demand variability pushes the hedging point $Z(L)$ upward.
Figure 8: Effect of the demand variability ($\mu_\text{H} = 1.5$, $\mu_\text{L} = 0.3$, $Ed = 0.9$, $\underline{u} = 1$, $A = 3$, $g^+ = 0.1$, $\eta = -3$, $M = 10$, $\gamma = 0.5$)
8 Conclusions

8.1 Summary

We have extended a widely-studied dynamic programming model of real-time scheduling control of manufacturing systems in two important ways: we model the effect of backlog on profits through an explicit representation of customer behavior; and we model random demand.

The new model of customer behavior involves a *defection function* which indicates what fraction of the potential customers choose not to complete their orders when the backlog reaches a given level. Because of this phenomenon, the model has a novel feature: the demand need not be less than capacity for there to exist a steady-state probability distribution of the inventory/backlog and the demand state.

We use a relationship with the lost sales problem to determine a solution structure, and we find that the solution involves a hedging point (to limit how far production should be allowed to go ahead of demand). To determine the hedging point, we find the steady-state probability distribution. We evaluate the objective function and choose values of the hedging point to maximize it.

Finally, we have performed a set of numerical experiments to demonstrate the behavior of the new model and the solution.

8.2 Future Research

This research can be extended in several different directions:

- Optimizing the hedging point could be greatly simplified by exploiting the equivalence between this model and the lost sales model mentioned in the Appendix. After evaluating the probability distribution once, using any hedging point, it is possible to construct the equivalent lost sales model and optimize it.

- Further understanding of the impact of the balking function could be gained by studying how it impacts a single parameter in an equivalent model. This parameter could be the stockout cost in the lost sales model (see Appendix) or the balking rate in a model with constant balking rate for all \( x < 0 \).

- An extension of the hedging point policy to complex systems (multiple part types; multiple stages; general routing including reentrant flow) is described by Gershwin (2000). In the present system, lead time is due only to the producer falling behind demand. In the more realistic system, lead time is also due to the fact that material flows from stage to stage, and may have to wait at each stage. The policy in Gershwin (2000) is based on a dynamic programming problem that includes an explicit backlog cost. It would be of interest to replace that backlog cost with the present model of customer behavior.

- The volume of sales that a business has should be a function of its past delivery performance. However, there is no way in the present model to account for the producer’s reputation for
on-time delivery. One way to include such an effect might be to add an appropriate state variable. For example, consider

\[ R^1(t) = \frac{\int_0^t u(\tau)d\tau}{\int_0^t d(\tau)d\tau} \]

This quantity is the average amount of demand actually served as a fraction of total potential demand. We can extend the demand model so that the demand parameters \((\mu_L, \mu_H, \lambda_{LH}, \lambda_{HL})\) are functions of \(R^1(t)\).

Another possible reputation variable is

\[ R^2(t) = \frac{\int_0^t (1 - B(x(\tau)))d\tau}{t} \]

which might be easier to include in the system dynamics. The integrand is the fraction of customers who do not defect.

- An important extension would be to include competition in the formulation, and turn it into a game. Now the \(B(x)\) function as seen by one firm depends on the actions taken by all competing firms.

- We have postulated the existence of the defection function \(B(x)\). It would be very desirable to use empirical data to understand the behavior of actual defection functions.

- In the Appendix of (Tan and Gershwin 2001), we analyze the effect of customer behavior on the customer defection function in a queueing model. This approach can be extended to analyze the effect of customer behavior, the capacity and the competitive position of the firm on customer defection.

Acknowledgments

We are grateful for support from the TUBITAK-NATO Science Fellowship program, the National Science Foundation, Grant DMI-9713500, the Lean Aircraft Initiative, and the Xerox Foundation.

Appendix—Proof of Theorem 1

First we show that the optimal policy is nonidling when there are backorders.
Lemma 2 \( w^*(x, D) = \underline{u} \) for \( x < 0 \).

**Proof.** Suppose \( u^*(x, D) < \underline{u} \) on a subset of \([x^*, 0]\) of nonzero measure. Consider the coupled processes \( x(t) \), \( x^1(t) \) with policies \( u^* \) and \( u^1 \) and identical initial conditions with \( x(0) < 0 \). Let \( u^1(x, D) = \underline{u} \) for \( x < 0 \), \( u^1(0, L) = \mu_L \) for \( t < \tau \equiv \min\{t : x(t) = 0\} \) so that \( x^1(t) \leq 0 \) until the processes merge at \( x(\tau) = x^1(\tau) = 0 \), and \( u^1(t) = u^*(t) \) for \( t > \tau \). Then

\[
E \int_0^\tau d(t)[1 - B(x(t))]dt < E \int_0^\tau d(t)[1 - B(x^1(t))]dt,
\]

i.e., \( x(t) \) experiences less demand, in expectation. The processes merge with probability one and the only cost difference is due to the difference in demand before merging:

\[
\lim_{\tau \to \infty} J^1(x, D; T) - J(x, D; T) = -E \int_0^\tau Ad(t)[B(x^1(t)) - B(x(t))]dt > 0,
\]

contradicting the optimality of \( u^* \).

Let \( Z(L) = \min\{x : u^*(x, L) < \underline{u}\} \). Because (8) is linear in \( u \), we can choose \( u^*(\cdot, L) \) to only have the values 0 and \( \underline{u} \), except at points where both maximize (8), where we can choose the value \( \mu_L \). Consequently, \( Z(L) \) is an upper bound: if \( x(0) \leq Z(L) \), then \( x(t) \leq Z(L) \).

If \( Z(L) = 0 \) we are done, so we can assume \( Z(L) > 0 \). Equation (45) can be written

\[
\mathcal{J} = A(Ed - \text{prob}[x \leq 0]\mu_B - g^+ \text{prob}[x > 0]E(x|x > 0) \tag{49}\]

where \( \mu_B = E(dB|x \leq 0) \) is the average rate at which customers balk when there is a backlog, and the stationary distribution is used. Because entry into \( x \leq 0 \) occurs at a single state \((0, H)\), the conditional distribution of \((x, D)\) given \( x \leq 0 \) is the same for all policies satisfying Lemma 1. In particular, \( \mu_B \) is the same.

We will establish a relationship between the backlog-dependent problem \( x(t) \) and the lost sales problem \( x^{LS}(t) \) of (Hu 1995). The dynamics can be written

\[
\frac{dx^{LS}}{dt} = u(t) + \underline{u} - \mu_H, \quad 0 \leq u(t) \leq \mu_H - \mu_L, D = L
\]

\[
geq 0, D = H \tag{50}
\]

and \( x^{LS}(t) \geq 0 \). On \( x > 0 \), (4)–(6) can be written

\[
\frac{dx}{dt} = u(x(t)) + \underline{u} - \mu_H, \quad (\mu_H - \mu_L) - \underline{u} \leq u(t) \leq \mu_H - \mu_L, D = L
\]

\[
-\underline{u} \leq u(t) \leq 0, D = H \tag{51}
\]

Thus, on \( x > 0 \) our model is a relaxation of \( x^{LS} \) in which the lower control limits are negative, where we have identified the operational state as \( L \), the failed state as \( H \), the demand rate \( \mu_H - \underline{u} \), and the production rate \( \mu_H - \mu_L \). Average cost for the lost sales problem is

\[
\mathcal{J}^{LS} = s \text{prob}[x^{LS} = 0] + g^+ E(x^{LS}|x^{LS} > 0) \text{prob}[x^{LS} > 0], \tag{52}
\]

26
where \( s \) is the stockout cost. Let \( Z(L)^{LS} \) be the optimal hedging point for \( x^{LS} \), and also assume \( Z(L)^{LS} > 0 \), so that \( \text{prob}[x^{LS} = 0, D = L] = 0 \) and (52) is equivalent to applying a cost to lost sales.

Suppose \( x \) and \( x^{LS} \) both use the optimal policy for \( x^{LS} \) on \( x > 0 \) and that \( x \) uses the optimal policy of Lemma 1 on \( x < 0 \). To compare their differential costs, set \( V_u(0; H) = V_{LS}(0; H) = 0 \). For some \( s \), \( V_u(0; L) = -V_{LS}(0; L) \), because \( V_{LS} \) is a continuous function of \( s \) (see also (16) of (Hu 1995)). Then, because they have the same dynamics on \( x > 0 \), entry into \( x > 0 \) occurs from a single state \((0; L)\), and departure from \( x > 0 \) occurs at the recurrent state \((0; H)\), their differential costs on \( x > 0 \) are the same:

\[
V_u(x, D) = -V_{LS}(x, D), \quad x > 0.
\] (53)

The negative sign is needed because \( x^{LS} \) is a minimization problem. Hu (1995) finds \( V_{LS} \) and shows that \( V_{LS}(\cdot; H) \) is decreasing for \( x < Z(H)^{LS} \), where \( Z(L)^{LS} < Z(H)^{LS} \). Thus, \( V_u(\cdot, D) \) is increasing and satisfies the Bellman equations (8) and (9) for \( 0 < x < Z(L)^{LS} \). By Lemma 1, \( V_u \) can be extended to also satisfy these equations for \( x \leq 0 \). Since \( V_u \) is fully specified, it must be the unique optimal differential cost, \( V(x, D) = V_u(x, D) \), for \( x < Z(L)^{LS} \).

Finally, \( V \) is continuously differentiable (see, e.g., (Sethi, Suo, Taksar, and Zhang 1997)) so \( \frac{dV}{dt}(Z(L)^{LS}, L) = \frac{dV_{LS}}{dt}(Z(L)^{LS}, L) = 0 \), and we can choose \( u^*(Z(L)^{LS}, L) = \mu_L \). That makes \( Z(L) = Z(L)^{LS} > 0 \) a hedging point for both problems and states \( x > Z(L) \) transient. Hence, the Bellman equation is satisfied in all recurrent states by the policy of Theorem 1, and it is optimal.

**Remark.** Another approach to Theorem 1 is to extend proofs of convexity of the differential cost, such as (Sethi, Suo, Taksar, and Zhang 1997), to show that \( V \) is concave. Concavity implies that the hedging point form extends to all states. However, the connection we have made with the lost sales problem makes explicit expressions for \( V \) available. The connection with the lost sales problem (or any problem with the same dynamics and cost rate on \( x > 0 \)) can also be seen by comparing (49) with (52) and recalling that \( \mu_B \) is the same for all policies satisfying Lemma 1. Of all policies with a given \( \text{prob}[x > 0] \), the optimal policy minimizes \( E(x|x > 0) \). In (52), again the optimal policy minimizes \( E(x^{LS}|x^{LS} > 0) \) for a given \( \text{prob}[x^{LS} > 0] \). When they use the same policy on the recurrent states \( x \leq Z(L) \), they have \( E(x|x > 0) = E(x^{LS}|x^{LS} > 0) \). Thus, policies have the same average cost ordering for both problems. This similarity demonstrates, as noted above, that our problem differs from the lost sales problem only in that the control limits are relaxed.

**References**


